Smart Materials, Adaptive Structures, and Intelligent Mechanical Systems

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Lecture 21
Analysis of a Laminated Composite
References for this Lecture


Lecture Overview

- Introduction

- Resultant Forces and Moments

- Piece-wise integration for calculating resultant forces and moments.
Introduction

• In previous lecture, mathematical relations have been developed, which define:
  – Variation of strains over the thickness of a laminate
  – Variation of stresses over the thickness of a laminate

• Given mid-plane strains and curvature, these relations may be used to calculate stresses in a plate.

• In a large number of real applications, we may not necessarily know mid-plane strains and curvatures for a composite, and sans their knowledge, predicting stresses in composite laminates is not possible just by using equations developed earlier.

• However, in a several cases we do know the value of externally applied loads and moments on plates. Thus, there is a need to develop relations which connect mid-plane strains, mid-plane curvatures, stresses, and external forces and moments.
Resultant Forces and Moments

- Consider a small part of composite plate, which is acted upon by forces and moments on its edges as shown in Fig. 21.1 due to different stresses. Here, \( N_x \), \( N_y \) and \( N_{xy} \) are \textit{resultant forces} per unit length acting on the edges of the composite plate in directions as shown in Fig. 21.1. Similarly, \( M_x \), \( M_y \) and \( M_{xy} \) are \textit{resultant moments} per unit length acting on the edges of the composite plate in directions as shown in Fig. 21.1.

Fig. 21.1: Forces and Moments at Mid-Plane of a Plate
Resultant Forces and Moments

- Using principles of equilibrium, we can relate stresses to force resultants by integrating appropriate stress components through the plate thickness. Thus, we get.

\[
N_x = \int_{-t/2}^{t/2} \sigma_{xx} \, dz \\
N_y = \int_{-t/2}^{t/2} \sigma_{yy} \, dz \\
N_{xy} = \int_{-t/2}^{t/2} \tau_{xy} \, dz
\]

(Eq. 21.1)

- In above equation, \( t \), represents the thickness of composite plate.
**Resultant Forces and Moments**

- Similarly, using principles of equilibrium, we can relate stresses to moment resultants by integrating appropriate products of stress components and distance from mid-plane, through the plate thickness. Thus, we get.

  \[
  M_x = \int_{-t/2}^{t/2} \sigma_{xx} z \, dz \\
  M_y = \int_{-t/2}^{t/2} \sigma_{yy} z \, dz \\
  M_{xy} = \int_{-t/2}^{t/2} \tau_{xy} z \, dz 
  \]

  (Eq. 21.3)

- In these equations, unit of force resultants (\(N_x\), \(N_y\) and \(N_{xy}\)) is N/m, while that for moment resultant (\(M_x\), \(M_y\) and \(M_{xy}\)) is N. Also, Fig. 21.1 depicts conventions used for positive resultant forces and resultant moments.
Resultant Forces and Moments

Fig. 21.2: Possible Variations in Stresses, Moduli and Strains Over the Thickness of a Hypothetical Four-Layer Laminate
Resultant Forces and Moments

• Consider Fig. 21.2. Looking at it, and also realizing from previous analysis, we infer that variation of stress is:
  – Discontinuous over the thickness of the whole laminate.
  – Linearly continuous over the thickness of a single layer. Such a variation of stresses over laminate thickness.

• Thus, the integrands for resultant forces and moments, as defined in Eqs. 21.1 and 21.2 are not continuous functions of \( z \).

• Rather, they are piece-wise continuous over the thickness of composite plate. Hence, their integration over entire thickness requires one to:
  – Piece-wise integrate the function over each lamina’s thickness.
  – Add up lamina specific integrals for all the layers.

  This is accomplished subsequently.
Resultant Forces and Moments

- Consider Fig. 21.3, which shows cross-section of the stack up of an $n$-layered orthotropic laminate. Here, the $z$ coordinate of top and bottom surfaces of $k^{th}$ laminate is $z_k$, and $z_{k-1}$. Further, as per the convention in this schematic, top most layer, with a $z$-coordinate of $t/2$ is considered the 1$^{st}$ layer, while the bottom most layer is considered the $n^{th}$ layer.

Fig. 21.3: Numbering Conventions for a Laminate with n Layers
Resultant Forces and Moments

- For such a schematic, the relations of resultant forces and moments, using Eqs. 21.1 and 21.2 can be written as:

\[
\begin{bmatrix}
N_x \\
N_y \\
N_{xy}
\end{bmatrix}
= \int_{-t/2}^{t/2} \begin{bmatrix}
\sigma_{xx} \\
\sigma_{yy} \\
\tau_{xy}
\end{bmatrix} dz
= \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} \begin{bmatrix}
\sigma_{xx}^k \\
\sigma_{yy}^k \\
\tau_{xy}^k
\end{bmatrix} dz
\]

(Eq. 21.4)

and,

\[
\begin{bmatrix}
M_x \\
M_y \\
M_{xy}
\end{bmatrix}
= \int_{-t/2}^{t/2} \begin{bmatrix}
\sigma_{xx} \\
\sigma_{yy} \\
\tau_{xy}
\end{bmatrix} zdz
= \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} \begin{bmatrix}
\sigma_{xx}^k \\
\sigma_{yy}^k \\
\tau_{xy}^k
\end{bmatrix} zdz
\]

(Eq. 21.5)
Resultant Forces and Moments

- Equations 21.4 and 21.5 are *summations* of integrals. If there are \( n \) layers in the composite, then there will be \( n \) summations.

- In this way, contribution of each layer is summed up while calculating resultant forces and moments.

- After integration and summation, coordinate \( z \) no longer appears in expressions for resultant forces and moments.

- These force resultants, \( N_x, N_y \), and \( N_{xy} \), and moment resultants, \( M_x, M_y, \) and \( M_{xy} \), get applied on a composite plate’s mid-plane, thereby generating stresses and strains in the plate.

- It should be noted here that even though these force and moment resultant do not vary with respect to the \( z \)-direction, they are indeed functions of \( x \) and \( y \) coordinates.