Smart Materials, Adaptive Structures, and Intelligent Mechanical Systems

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Lecture 25
Semi-Infinite Plates
References for this Lecture


Lecture Overview

• Solution for Case C

• Solution for Case D
Solution for Case C

- First, we reproduce equilibrium relations (Eq. 24.2), which apply to all cases. These are:
  \[ N_x = c_1 \]
  \[ N_{xy} = c_2 \]
  \[ M_x = -qx^2/2 + c_3x + c_4 \]  
  (Eq. 24.2)

- Further, we also reproduce Eq. 24.1 since it represents kinematic relations for all cases.

\[
\begin{align*}
\{\varepsilon_{xx}\} & = \begin{pmatrix} \varepsilon^0_{xx} \\ \varepsilon^0_{yy} \end{pmatrix} + z \begin{pmatrix} k_x \\ k_y \\ k_{xy} \end{pmatrix} \\
\{\gamma_{xy}\} & = \begin{pmatrix} d^u \\ 0 \\ d^v \end{pmatrix} \\
\end{align*}
\]

where,

\[
\begin{align*}
\{\varepsilon^0_{xx}\} & = \begin{pmatrix} \frac{du^0}{dx} \\ 0 \\ \frac{dv^0}{dx} \end{pmatrix} \\
\{\gamma^0_{xy}\} & = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\
\end{align*}
\]

and

\[
\begin{align*}
\{k_x\} & = \begin{pmatrix} \frac{d^2w^0}{dx^2} \\ 0 \\ 0 \end{pmatrix} \\
\end{align*}
\]
Solution for Case C

- For Cases C and D, the lamination sequence is \([0_2/90_2]_\text{T}\). For such a laminate, we reproduce results from Eq. 24.4.
  - \(A_{16} = A_{26} = 0\)
  - \(B_{16} = B_{26} = B_{12} = B_{66} = 0\)
  - \(D_{16} = D_{26} = 0\)  
(Eq. 24.4)

- Finally, boundary conditions for Case C are:

<table>
<thead>
<tr>
<th>BCs at (x = -a/2)</th>
<th>BCs at (x = +a/2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(u^\circ\cdot = 0)</td>
<td>(N_x^+ = 0)</td>
</tr>
<tr>
<td>(v^\circ\cdot = 0)</td>
<td>(N_{xy}^+ = 0)</td>
</tr>
<tr>
<td>(M_x^- = 0)</td>
<td>(M_x^+ = 0)</td>
</tr>
<tr>
<td>(w^\circ\cdot = 0)</td>
<td>(w^{o+} = 0)</td>
</tr>
</tbody>
</table>

- Since \(N_x\) is zero at \(+a/2\), we get from Eq. 20.2:
  \[C_1 = N_x = 0 = A_{11}(du^\circ/dx) + B_{11}(d^2w^\circ/dx^2),\]
or,
  \[u^\circ(x) = [C_1x + C_5 + B_{11}(dw^\circ/dx)]/A_{11}\]  
(Eq. 25.1)
Solution for Case C

• Since $N_{xy}$ is zero at $+\alpha/2$, we get from earlier results:

\[
C_2 = N_{xy} = 0 = A_{66}(dv^o/dx), \text{ or,}
\]
\[
v^o(x) = [C_2x + C_6]/A_{66} \quad \text{(Eq. 25.2)}
\]

• Also, we know that $M_x = B_{11}(du^o/dx) + D_{11}(d^2w^o/dx^2)$. Integrating this equation after substituting the definition of $M_x$ from Eq. 16.2, we get:

\[
B_{11}u^o(x) + D_{11}(dw^o/dx) = -qx^3/6 + C_3x^2/2 + C_4x + C_7 \quad \text{(Eq. 25.3)}
\]

• Equations 25.1-3 can be used to solve for $u^o$, $v^o$, and $w^o$. Here, the term $B_{11}$ couples in-plane and out-of-plane displacements. Solving Eqs. 25.1-3 for displacement field yields:

\[
(A_{11} - B_{11}^2/D_{11}) u^o(x) = (B_{11}/D_{11})(qx^3/6) - (B_{11}/A_{11})(C_3qx^2/2) + (C_1 - C_4B_{11}/A_{11})x + (C_5 - C_7B_{11}/A_{11})
\]

\[
(D_{11} - B_{11}^2/A_{11}) w^o(x) = qx^4/24 - C_3x^3/6 - (C_1B_{11}/A_{11} - C_4)x^2/2 + (C_5B_{11}/A_{11} - C_7)x + C_8
\]

(Eq. 25.4a, b)
Solution for Case C

- Also, \( M_y = D_{12} \left( \frac{d^2 w^o}{dx^2} \right) \), and \( M_{xy} = 0 \) (Eq. 25.5)

- Equations 25.1-25.5 are applicable to Case C and Case D, as BCs yet remain to be applied. At this stage, we apply boundary conditions for Case C.

- Since \( N_x \) and \( N_{xy} \) are zero at \( x = +a/2 \), we get:
  \[ C_1 = C_2 = 0. \]
- Further, since \( v^o \) is zero at \( x = +a/2 \), we get from Eq. 25.2:
  \[ C_6 = 0. \]
- Further, since moment \( M_x \) is zero at both ends, we get from Eq. 20.2:
  \[ C_3 = 0 \text{ and } C_4 = qa^2/8. \]
- Now, we are left with \( C_5 \), \( C_7 \), and \( C_8 \). To find them, we apply two BCs for \( w^o \) condition, and one BC for the \( u^o \) condition.
Solution for Case C

• Applying two BCs for $w^o$ in Eq. 25.4b, and one BC for $u^o$ in Eq. 25.4a, we get values for $C_5$, $C_7$, and $C_8$. Finally, we get the following relations for $u^o$, $v^o$, and $w^o$:

\[
 u^o(x) = \left( B_{11}qa^3 \right) \left[ 4(x/a)^3 - 3(x/a) - 1 \right] / \left[ 24D_{11}(A_{11} - B_{11}^2/D_{11}) \right] \quad \text{(Eq. 25.6)}
\]

\[
 v^o(x) = 0 \quad \text{(Eq. 25.7)}
\]

\[
 w^o(x) = \left( -5qa^4 \right) \left[ 16(x/a)^4 - 24(x/a)^2 + 5 \right] / \left[ 384(D_{11} - B_{11}^2/A_{11}) \right] \quad \text{(Eq. 25.8)}
\]

• Equations 18.6-8, constitute the displacement-field for Case C.

• Next, we consider Case D. Here, the general solution as expressed through Eqs. 25.1-5 is also the valid for Case D. However, integration constants are different due to differences in boundary conditions.
Solution for Case D

• The boundary conditions for Case D are given below.

<table>
<thead>
<tr>
<th>BCs at x = -a/2</th>
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</thead>
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<tr>
<td>$u^{o-} = 0$</td>
<td>$u^{o+} = 0$</td>
</tr>
<tr>
<td>$v^{o-} = 0$</td>
<td>$v^{o+} = 0$</td>
</tr>
<tr>
<td>$M_{x}^{o-} = 0$</td>
<td>$M_{x}^{o+} = 0$</td>
</tr>
<tr>
<td>$w^{o-} = 0$</td>
<td>$w^{o+} = 0$</td>
</tr>
</tbody>
</table>

• Using these boundary conditions we get the final solutions as:

\[ u^{o}(x) = (B_{11}qa^{3})[4(x/a)^2 - 1]/ [24D_{11}(A_{11} - B_{11}^{2}/D_{11})](x/a) \]  
\[ \text{(Eq. 25.9)} \]

\[ v^{o}(x) = 0 \]  
\[ \text{(Eq. 25.10)} \]

\[ w^{o}(x) = (-qa^4)[16(x/a)^4 + 48B_{11}/(3A_{11}D_{11}) - 1/2](x/a)^2 + \{5 - 4B_{11}^2/(A_{11}D_{11})\}/ [384 (D_{11} - B_{11}^2/A_{11})] \]  
\[ \text{(Eq. 25.11)} \]
Comments on Cases C & D

- In Cases C and D, the direction of \( u^o(x) \) depends on sign of \( B_{11} \).
  - The sign for \( B_{11} \) for a \([0_2/90_2]_T\) laminate is negative of that for a \([90_2/0_2]_T\) laminate.
  - Thus, direction of \( u^o(x) \) for a \([0_2/90_2]_T\) laminate is negative of that for a \([90_2/0_2]_T\) laminate.

- In either case, \( u^o(x) \) and \( w^o(x) \) are odd and even functions of \( x \). This observation is consistent with intuition.

- The term \( (D_{11}-B_{11}^2/A_{11}) \) is known as reduced bending stiffness. It appears as denominator in expressions for \( w^o(x) \). The out of plane displacement is inversely proportional to this term, and not just \( D_{11} \). For symmetric laminates, this term is identical to \( D_{11} \). Also, higher the value of \( B_{11} \), lower reduced bending stiffness.

- The term \( (A_{11}-B_{11}^2/D_{11}) \) is known as reduced extensional stiffness. It appears as denominator in expressions for \( u^o(x) \). The in-plane displacement is inversely proportional to this term, and not just \( A_{11} \).