Module
2
Stresses in machine elements

Lesson

3

Strain analysis

Instructional Objectives

At the end of this lesson, the student should learn

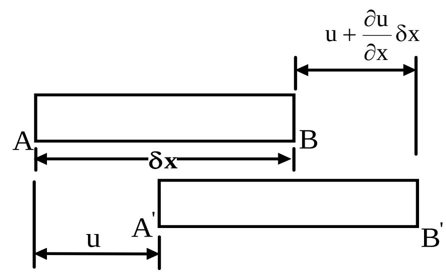
- Normal and shear strains.
- 3-D strain matrix.
- Constitutive equation; generalized Hooke's law
- Relation between elastic, shear and bulk moduli (E, G, K).
- Stress- strain relation considering thermal effects.

2.3.1 Introduction

No matter what stresses are imposed on an elastic body, provided the material does not rupture, displacement at any point can have only one value. Therefore the displacement at any point can be completely given by the three single valued components u, v and w along the three co-ordinate axes x, y and z respectively. The normal and shear strains may be derived in terms of these displacements.

2.3.2 Normal strains

Consider an element AB of length δx (figure-2.3.2.1). If displacement of end A is u, that of end B is $u + \frac{\partial u}{\partial x}\delta x$. This gives an increase in length of ($u + \frac{\partial u}{\partial x}\delta x$ -u) and therefore the strain in x-direction is $\frac{\partial u}{\partial x}$.Similarly, strains in y and z directions are $\frac{\partial v}{\partial y}$ and $\frac{\partial w}{\partial z}$.Therefore, we may write the three normal strain components as $\varepsilon_x = \frac{\partial u}{\partial x}$, $\varepsilon_y = \frac{\partial v}{\partial y}$ and $\varepsilon_z = \frac{\partial w}{\partial z}$.

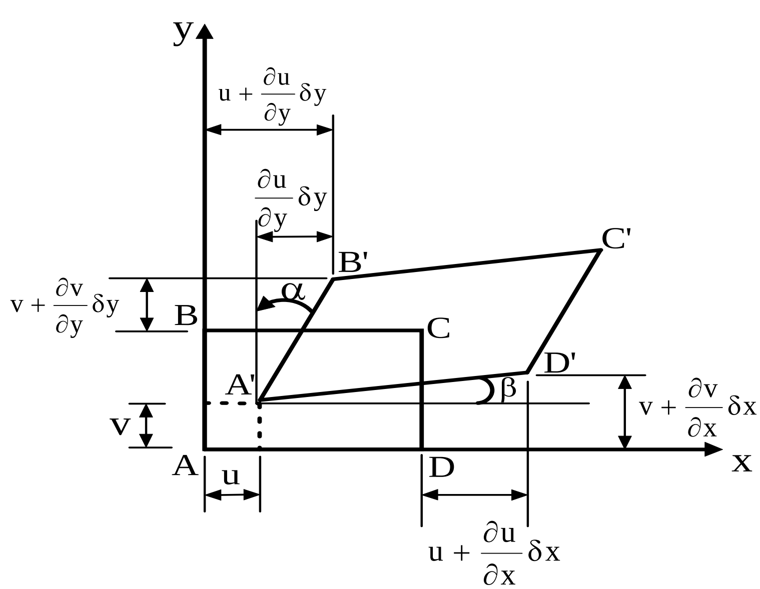


***2.3.2.1F**- Change in length of an infinitesimal element.*

2.3.3 Shear strain

In the same way we may define the shear strains. For this purpose consider an element ABCD in x-y plane and let the displaced position of the element be A′B′C′D′ (Figure-2.3.3.1). This gives shear strain in xy plane as $\varepsilon_{xy} = \alpha + \beta$ where α is the angle made by the displaced line B′C′ with the vertical and β is the angle made by the displaced line A′D′ with the horizontal. This gives

$$\alpha = \frac{\frac{\partial u}{\partial y}\delta y}{\delta y} = \frac{\partial u}{\partial y} \quad \text{and} \quad \beta = \frac{\frac{\partial v}{\partial x}\delta x}{\delta x} = \frac{\partial v}{\partial x}$$



***2.3.3.1F**- Shear strain associated with the distortion of an infinitesimal element.*

We may therefore write the three shear strain components as

$$\varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \ \varepsilon_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \ \text{and} \ \varepsilon_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

Therefore, the complete strain matrix can be written as

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \varepsilon_{xy} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0 \\ 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \end{Bmatrix}$$

2.3.4 Constitutive equation

The state of strain at a point can be completely described by the six strain components and the strain components in their turns can be completely defined by the displacement components u, v, and w. The constitutive equations relate stresses and strains and in linear elasticity we simply have σ=Eε where E is modulus of elasticity. It is also known that σ_x produces a strain of σ_x/E in x-direction, $-\nu\sigma_x/E$ in y-direction and $-\nu\sigma_x/E$ in z-direction . Therefore we may write the generalized Hooke's law as

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu(\sigma_y + \sigma_z)\right], \ \ \varepsilon_y = \frac{1}{E}\left[\sigma_y - \nu(\sigma_z + \sigma_x)\right] \ \text{and} \ \varepsilon_z = \frac{1}{E}\left[\sigma_z - \nu(\sigma_x + \sigma_y)\right]$$

It is also known that the shear stress $\tau = G\gamma$, where G is the shear modulus and γ is shear strain. We may thus write the three strain components as

$$\gamma_{xy} = \frac{\tau_{xy}}{G}, \ \gamma_{yz} = \frac{\tau_{yz}}{G} \ \text{and} \ \gamma_{zx} = \frac{\tau_{zx}}{G}$$

In general each strain is dependent on each stress and we may write

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} \\ K_{21} & K_{22} & K_{23} & K_{24} & K_{25} & K_{26} \\ K_{31} & K_{32} & K_{33} & K_{34} & K_{35} & K_{36} \\ K_{41} & K_{42} & K_{43} & K_{44} & K_{45} & K_{46} \\ K_{51} & K_{52} & K_{53} & K_{54} & K_{55} & K_{56} \\ K_{61} & K_{62} & K_{63} & K_{64} & K_{65} & K_{66} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix}$$

For isotropic material

$$K_{11} = K_{22} = K_{33} = \frac{1}{E}$$

$$K_{12} = K_{13} = K_{21} = K_{23} = K_{31} = K_{32} = -\frac{\nu}{E}$$

$$K_{44} = K_{55} = K_{66} = \frac{1}{G}$$

Rest of the elements in K matrix are zero.

On substitution, this reduces the general constitutive equation to equations for isotropic materials as given by the generalized Hooke's law. Since the principal stress and strains axes coincide, we may write the principal strains in terms of principal stresses as

$$\varepsilon_1 = \frac{1}{E}\left[\sigma_1 - \nu(\sigma_2 + \sigma_3)\right]$$

$$\varepsilon_2 = \frac{1}{E}\left[\sigma_2 - \nu(\sigma_3 + \sigma_1)\right]$$

$$\varepsilon_3 = \frac{1}{E}\left[\sigma_3 - \nu(\sigma_1 + \sigma_2)\right]$$

From the point of view of volume change or dilatation resulting from hydrostatic pressure we also have

$$\bar{\sigma} = K\Delta$$

where $\bar{\sigma} = \frac{1}{3}\left(\sigma_x + \sigma_y + \sigma_z\right) = \frac{1}{3}\left(\sigma_1 + \sigma_2 + \sigma_3\right)$ and $\Delta = (\varepsilon_x + \varepsilon_y + \varepsilon_z) = (\varepsilon_1 + \varepsilon_2 + \varepsilon_3)$

These equations allow the principal strain components to be defined in terms of principal stresses. For isotropic and homogeneous materials only two constants viz. E and ν are sufficient to relate the stresses and strains.

The strain transformation follows the same set of rules as those used in stress transformation except that the shear strains are halved wherever they appear.

2.3.5 Relations between E, G and K

The largest maximum shear strain and shear stress can be given by

$\gamma_{max} = \varepsilon_2 - \varepsilon_3$ and $\tau_{max} = \frac{\sigma_2 - \sigma_3}{2}$ and since $\gamma_{max} = \frac{\tau_{max}}{G}$ we have

$\frac{1}{E}\left[\sigma_2 - \nu\left(\sigma_1 + \sigma_3\right)\right] - \frac{1}{E}\left[\sigma_3 - \nu\left(\sigma_1 + \sigma_2\right)\right] = \frac{1}{G}\left(\frac{\sigma_2 - \sigma_3}{2}\right)$ and this gives

$$G = E/2(1+\nu)$$

Considering now the hydrostatic state of stress and strain we may write

$\frac{1}{3}\left(\sigma_1 + \sigma_2 + \sigma_3\right) = K(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)$. Substituting ε_1, ε_2 and ε_3 in terms of σ_1 , σ_2 and σ_3 we may write

$\frac{1}{3}\left(\sigma_1 + \sigma_2 + \sigma_3\right) = K\left[(\sigma_1 + \sigma_2 + \sigma_3) - 2\nu(\sigma_1 + \sigma_2 + \sigma_3)\right]$ and this gives

$$K = E/3(1-2\nu).$$

2.3.6 Elementary thermoelasticity

So far the state of strain at a point was considered entirely due to applied forces. Changes in temperature may also cause stresses if a thermal gradient or some external constraints exist. Provided that the materials remain linearly elastic, stress pattern due to thermal effect may be superimposed upon that due to applied forces and we may write

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu(\sigma_y + \sigma_z)\right] + \alpha T \qquad \varepsilon_{xy} = \frac{\tau_{xy}}{G}$$

$$\varepsilon_y = \frac{1}{E}\left[\sigma_y - \nu(\sigma_z + \sigma_x)\right] + \alpha T \quad \text{and} \quad \varepsilon_{yz} = \frac{\tau_{yz}}{G}$$

$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - \nu(\sigma_x + \sigma_y)\right] + \alpha T \qquad \varepsilon_{zx} = \frac{\tau_{zx}}{G}$$

It is important to note that the shear strains are not affected directly by temperature changes. It is sometimes convenient to express stresses in terms of strains. This may be done using the relation $\Delta = \varepsilon_x + \varepsilon_y + \varepsilon_z$. Substituting the above expressions for ε_x, ε_y and ε_z we have,

$$\Delta = \frac{1}{E}\left[(1-2\nu)\left(\sigma_x + \sigma_y + \sigma_z\right)\right] + 3\alpha T$$

and substituting $K = E/3(1-2\nu)$ we have

$$\Delta = \frac{1}{3K}\left(\sigma_x + \sigma_y + \sigma_z\right) + 3\alpha T .$$

Combining this with $\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu(\sigma_y + \sigma_z)\right] + \alpha T$ we have

$$\sigma_x = \frac{E\varepsilon_x}{1+\nu} + \frac{3\nu K(\Delta - 3\alpha T)}{1+\nu} - \frac{E\alpha T}{1+\nu}$$

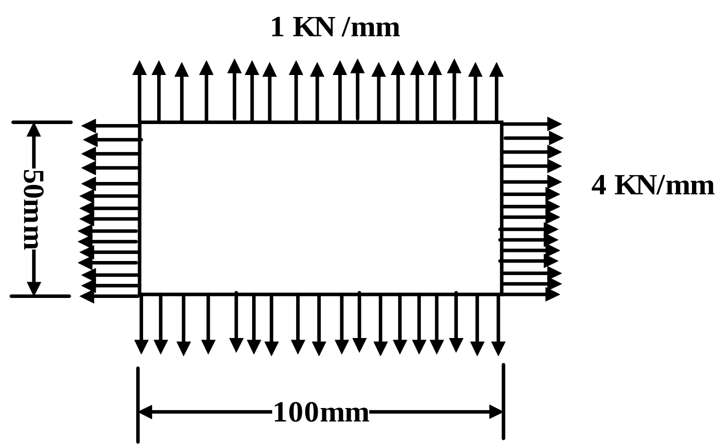
Substituting $G = E/2(1+\nu)$ and $\lambda = \frac{3\nu K}{1+\nu}$ we may write the normal and shear stresses as

$$\sigma_x = 2G\varepsilon_x + \lambda\Delta - 3K\alpha T$$
$$\sigma_y = 2G\varepsilon_y + \lambda\Delta - 3K\alpha T$$
$$\sigma_z = 2G\varepsilon_z + \lambda\Delta - 3K\alpha T$$
$$\tau_{xy} = G\varepsilon_{xy}$$
$$\tau_{yz} = G\varepsilon_{yz}$$
$$\tau_{zx} = G\varepsilon_{zx}$$

These equations are considered to be suitable in thermoelastic situations.

2.3.7 Problems with Answers

Q.1: A rectangular plate of 10mm thickness is subjected to uniformly distributed load along its edges as shown in figure-2.3.7.1. Find the change in thickness due to the loading. E=200 GPa, ν = 0.3



2.3.7.1F

A.1: Here σ_x = 400 MPa, σ_y = 100 MPa and σ_z = 0

This gives $\varepsilon_z = -\frac{\nu}{E}\left(\sigma_x + \sigma_y\right) = -7.5x10^{-4}$

Now, $\varepsilon_z = \frac{\Delta t}{t}$ where, t is the thickness and Δt is the change in thickness.

Therefore, the change in thickness = 7.5 μm.

Q.2: At a point in a loaded member, a state of plane stress exists and the strains are ε_x= -90x10^{-6}, ε_y= -30x10^{-6} and ε_{xy}=120x10^{-6}. If the elastic constants E , ν and G are 200 GPa , 0.3 and 84 GPa respectively, determine the normal stresses σ_x and σ_y and the shear stress τ_{xy} at the point.

A.2:

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu\sigma_y\right]$$

$$\varepsilon_y = \frac{1}{E}\left[\sigma_y - \nu\sigma_x\right]$$

$$\varepsilon_{xy} = \frac{\tau_{xy}}{G}$$

This gives $\sigma_x = \frac{E}{1-\nu^2}\left[\varepsilon_x + \nu\varepsilon_y\right]$

$$\sigma_y = \frac{E}{1-\nu^2}\left[\varepsilon_y + \nu\varepsilon_x\right]$$

Substituting values, we get

σ_x = -21.75 MPa, σ_y = -12.53 MPa and τ_{xy} = 9.23 MPa.

Q.3: A rod 50 mm in diameter and 150 mm long is compressed axially by an uniformly distributed load of 250 KN. Find the change in diameter of the rod if E = 200 GPa and ν=0.3.

A.3:

Axial stress $\sigma_x = \frac{250}{\frac{\pi}{4}(0.05)^2} = 127.3\,\text{MPa}$

Axial strain, $\varepsilon_x = 0.636\text{x}10^{-3}$

Lateral strain = $\nu\varepsilon_x = 1.9\text{x}10^{-4}$

Now, lateral strain, $\varepsilon_L = \frac{\Delta}{D}$ and this gives

Δ= 9.5 μm.

Q.4: If a steel rod of 50 mm diameter and 1m long is constrained at the ends and heated to 200^{o}C from an initial temperature of 20^{o}C, what would be the axial load developed? Will the rod buckle? Take the coefficient of thermal expansion, α=12x10^{-6} per ^{o}C and E=200 GPa.

A.4:

Thermal strain, $\varepsilon_t = \alpha \Delta T = 2.16x10^{-3}$

In the absence of any applied load, the force developed due to thermal expansion, $F = E\varepsilon_t A = 848KN$

For buckling to occur the critical load is given by

$$F_{cr} = \frac{\pi^2 EI}{l^2} = 605.59\,KN.$$

Therefore, the rod will buckle when heated to 200^{o}C.

2.3.8 Summary of this Lesson

Normal and shear strains along with the 3-D strain matrix have been defined. Generalized Hooke's law and elementary thermo-elasticity are discussed.

2.3.9 Reference for Module-2

1) Mechanics of materials by E.P.Popov, Prentice hall of India, 1989.
2) Mechanics of materials by Ferdinand P. Boer, E. Russel Johnson, J.T Dewolf, Tata McGraw Hill, 2004.
3) Advanced strength and applied stress analysis by Richard G. Budyens, McGraw Hill, 1999.
4) Mechanical engineering design by Joseph E. Shigley, McGraw Hill, 1986.