Lecture – 3
Classical Control Overview – I

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Review of Laplace Transforms

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Laplace Transform

Laplace Transform of $f(t)$:

$$F(s) = \int_{0^-}^{\infty} f(t) e^{-st} dt$$

($s = \sigma + j\omega$ : a complex variable)

Inverse Laplace Transform of $F(s)$:

$$L^{-1}[F(s)] = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} F(s) e^{st} ds$$

$$= f(t)u(t) \quad \text{where } u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$
Test Signals Commonly Used in Control Systems

<table>
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<th>Function</th>
<th>Description</th>
<th>Sketch</th>
<th>Use</th>
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<td>Impulse</td>
<td>δ(t)</td>
<td>δ(t) = 0 for 0⁻ &lt; t &lt; 0⁺  = 0 elsewhere</td>
<td><img src="image" alt="Impulse Sketch" /></td>
<td>Transient response modeling</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \int_{0^-}^{0^+} \delta(t) , dt = 1 )</td>
<td></td>
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<tr>
<td>Step</td>
<td>u(t)</td>
<td>u(t) = 1 for t &gt; 0  = 0 for t &lt; 0</td>
<td><img src="image" alt="Step Sketch" /></td>
<td>Transient response steady-state error</td>
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<td>Ramp</td>
<td>m(t)</td>
<td>m(t) = t for t ≥ 0  = 0 elsewhere</td>
<td><img src="image" alt="Ramp Sketch" /></td>
<td>Steady-state error</td>
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<td>Parabola</td>
<td>t²/2u(t)</td>
<td>t²/2u(t) = t² for t ≥ 0  = 0 elsewhere</td>
<td><img src="image" alt="Parabola Sketch" /></td>
<td>Steady-state error</td>
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<td>Sinusoid</td>
<td>sin(ωt)</td>
<td>sin(ωt)</td>
<td><img src="image" alt="Sinusoid Sketch" /></td>
<td>Transient response modeling steady-state error</td>
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</table>

Example – 1

\[ L(t^n) = \int_0^\infty e^{-st} t^n \, dt \quad \text{(by definition)} \]

Let \( v = st \) \( \Rightarrow \) \( dv = s \, dt \)

\[ L(t^n) = \int_0^\infty e^{-v} \left( \frac{v}{s} \right)^n \frac{dv}{s} \]

\[ = \frac{1}{s^{n+1}} \int_0^\infty e^{-v} v^n \, dv = \frac{n!}{s^{n+1}} \]

\( = n! \) \( \text{(by induction)} \)
Example – 2

\[ L(e^t) = \int_{0}^{\infty} e^{-st} e^t \, dt \quad (\text{by definition}) \]

\[ = \int_{0}^{\infty} e^{-(s-1)t} \, dt \]

\[ = \left[ \frac{e^{-(s-1)t}}{-(s-1)} \right]_{0}^{\infty} = -\frac{1}{(s-1)}[0 - 1] \]

\[ = \frac{1}{(s-1)} \]
### Laplace Transform

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<th>$f(t)$</th>
<th>$F(s)$</th>
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<td>1.</td>
<td>$\delta(t)$</td>
<td>1</td>
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<tr>
<td>2.</td>
<td>$u(t)$</td>
<td>$\frac{1}{s}$</td>
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<tr>
<td>3.</td>
<td>$tu(t)$</td>
<td>$\frac{1}{s^2}$</td>
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<td>4.</td>
<td>$t^n u(t)$</td>
<td>$\frac{n!}{s^{n+1}}$</td>
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<td>5.</td>
<td>$e^{-at} u(t)$</td>
<td>$\frac{1}{s + a}$</td>
</tr>
<tr>
<td>6.</td>
<td>$\sin \omega t u(t)$</td>
<td>$\frac{\omega}{s^2 + \omega^2}$</td>
</tr>
<tr>
<td>7.</td>
<td>$\cos \omega t u(t)$</td>
<td>$\frac{s}{s^2 + \omega^2}$</td>
</tr>
</tbody>
</table>

**Ref:** N. S. Nise: Control Systems Engineering, 4th Ed., Wiley, 2004
## Laplace Transform

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<th>Item no.</th>
<th>Theorem</th>
<th>Name</th>
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<td>( \mathcal{L}[f(t)] = F(s) = \int_0^\infty f(t)e^{-st}dt )</td>
<td>Definition</td>
</tr>
<tr>
<td>2.</td>
<td>( \mathcal{L}[kf(t)] = kF(s) )</td>
<td>Linearity theorem</td>
</tr>
<tr>
<td>3.</td>
<td>( \mathcal{L}[f_1(t) + f_2(t)] = F_1(s) + F_2(s) )</td>
<td>Linearity theorem</td>
</tr>
<tr>
<td>4.</td>
<td>( \mathcal{L}[e^{-at}f(t)] = F(s + a) )</td>
<td>Frequency shift theorem</td>
</tr>
<tr>
<td>5.</td>
<td>( \mathcal{L}[f(t - T)] = e^{-sT}F(s) )</td>
<td>Time shift theorem</td>
</tr>
<tr>
<td>6.</td>
<td>( \mathcal{L}[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right) )</td>
<td>Scaling theorem</td>
</tr>
<tr>
<td>7.</td>
<td>( \mathcal{L}\left[\frac{df}{dt}\right] = sF(s) - f(0-) )</td>
<td>Differentiation theorem</td>
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<tr>
<td>8.</td>
<td>( \mathcal{L}\left[\frac{d^2f}{dt^2}\right] = s^2F(s) - sf(0-) - \dot{f}(0-) )</td>
<td>Differentiation theorem</td>
</tr>
<tr>
<td>9.</td>
<td>( \mathcal{L}\left[\frac{d^nf}{dt^n}\right] = s^nF(s) - \sum_{k=1}^{n} s^{n-k}f^{k-1}(0-) )</td>
<td>Differentiation theorem</td>
</tr>
<tr>
<td>10.</td>
<td>( \mathcal{L}\left[\int_0^t f(\tau) d\tau\right] = \frac{F(s)}{s} )</td>
<td>Integration theorem</td>
</tr>
<tr>
<td>11.</td>
<td>( f(\infty) = \lim_{{s \to 0}} sF(s) )</td>
<td>Final value theorem(^1)</td>
</tr>
<tr>
<td>12.</td>
<td>( f(0+) = \lim_{{s \to \infty}} sF(s) )</td>
<td>Initial value theorem(^2)</td>
</tr>
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</table>

\(^1\) For this theorem to yield correct finite results, all roots of the denominator of \( F(s) \) must have negative real parts and no more than one can be at the origin.

\(^2\) For this theorem to be valid, \( f(t) \) must be continuous or have a step discontinuity at \( t = 0 \) (i.e., no impulses or their derivatives at \( t = 0 \)).
Result:

\[ L\left[ e^{-at} f(t) \right] = F(s + a) \]

\[
L\left[ e^{-at} f(t) \right] = \int_0^\infty e^{-st} e^{-at} f(t) \, dt = \int_0^\infty e^{-(s+a)t} f(t) \, dt
\]

Let \( \hat{s} = s + a \)

\[
L\left[ e^{-at} f(t) \right] = \int_0^\infty e^{-\hat{s}t} f(t) \, dt
\]

\[ = F(\hat{s}) \quad \text{(by definition)} \]

\[ = F(s + a) \]
Examples

(1) We know: \( L(\sin 2t) = \frac{2}{s^2 + 2^2} \)

Hence \( L(e^{-3t} \sin 2t) = \frac{2}{(s + 3)^2 + 2^2} = \frac{2}{s^2 + 6s + 13} \)

(2) We know: \( L(\cos 2t) = \frac{s}{s^2 + 2^2} \)

Hence \( L(e^{-3t} \cos 2t) = \frac{s + 3}{(s + 3)^2 + 2^2} = \frac{s + 3}{s^2 + 6s + 13} \)
Result:

\[ L\left[ t^n f(t) \right] = (-1)^n \frac{d^n F(s)}{ds^n} \]

By definition \( F(s) = \int_{0}^{\infty} e^{-st} f(t) dt \)

Hence \( \frac{dF(s)}{ds} = \frac{d}{ds} \int_{0}^{\infty} e^{-st} f(t) dt = \int_{0}^{\infty} \frac{d}{ds} \left[ e^{-st} f(t) \right] dt \)

\[ = \int_{0}^{\infty} -te^{-st} f(t) dt \]

\[ = (-1)^n \int_{0}^{\infty} e^{-st} \left[ t f(t) \right] dt \]

\[ = (-1)^n L \left[ t f(t) \right] \]
Result: 

\[ L \left[ t^n f(t) \right] = (-1)^n \frac{d^n F(s)}{ds^n} \]

Hence 

\[ L \left[ t f(t) \right] = (-1) \frac{dF(s)}{ds} \]

Similarly 

\[ L \left[ t^2 f(t) \right] = (-1)^2 \frac{d^2 F(s)}{ds^2} \]

\[ \vdots \]

\[ \vdots \]

In general 

\[ L \left[ t^n f(t) \right] = (-1)^n \frac{d^n F(s)}{ds^n} \]
Result:

\[
L \left[ \frac{df(t)}{dt} \right] = sF(s) - f(0)
\]

\[
L \left[ \frac{df(t)}{dt} \right] = \int_0^\infty e^{-st} \frac{df(t)}{dt} \, dt
\]

\[
= \left[ e^{-st} f(t) \right]_0^\infty - \int_0^\infty (-s) e^{-st} f(t) \, dt
\]

\[
= \left[ 0 - f(0) \right] + s \int_0^\infty e^{-st} f(t) \, dt
\]

\[
= s \left\{ F(s) \right\} - f(0)
\]

\[
= 0 \text{ (Typically)}
\]

Hence, multiplication by \( s \) is a derivative operator!
Generalization

\[
L \left[ \frac{d^2 f(t)}{dt^2} \right] = L \left[ \frac{d}{dt} \left( \frac{df(t)}{dt} \right) \right] = s \left[ s F(s) - f(0) \right] - f'(0)
\]

\[
= s^2 F(s) - s f(0) - f'(0)
\]

\[
L \left[ \frac{d^3 f(t)}{dt^3} \right] = s \left[ s^2 F(s) - s f(0) - f'(0) \right] - f''(0)
\]

\[
= s^3 F(s) - s^2 f(0) - s f'(0) - f''(0)
\]
Result: \[ L\left[ \int_{0}^{t} f(\tau) d\tau \right] = \frac{1}{s} F(s) \]

Let \( g(t) = \int_{0}^{t} f(\tau) d\tau \)

Then \( g(0) = 0, \quad g'(t) = f(t) \)

\[ F(s) = L[f(t)] = L[g'(t)] = sL[g(t)] - g(0) = s L\left[ \int_{0}^{t} f(\tau) d\tau \right] \]

Hence \[ L\left[ \int_{0}^{t} f(\tau) d\tau \right] = \frac{1}{s} F(s) \]

i.e. Division by \( s \) is an integral operator!
Transfer Function Representation

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Block Diagram Representation

\[ R(s) = \frac{(b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0)}{(a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0)} \]

\[ C(s) \]
Transfer Function Representation

Any physical system that can be represented by a linear, time-invariant constant coefficient differential equation can be modeled as a Transfer function

\[
a_n \frac{d^n c(t)}{dt^n} + a_{n-1} \frac{d^{n-1} c(t)}{dt^{n-1}} + \cdots + a_0 c(t)
\]

\[
= b_m \frac{d^m r(t)}{dt^m} + b_{m-1} \frac{d^{m-1} r(t)}{dt^{m-1}} + \cdots + b_0 r(t)
\]

\(c(t)\): the output \(r(t)\): the input

\(a_i\)'s and \(b_i\)'s are constants
Transfer Function Representation

- Taking Laplace Transform
  \[ a_n s^n C(s) + a_{n-1} s^{n-1} C(s) + \cdots \]
  \[ \cdots + a_0 C(s) + \text{initial condition terms} \]
  \[ = b_m s^m R(s) + b_{m-1} s^{m-1} R(s) + \cdots \]
  \[ \cdots + b_0 R(s) + \text{initial condition terms} \]

- Assume all initial conditions as zero (linear system)
  Then the ratio
  \[ T(s) = \frac{C(s)}{R(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0} \]
  is called the **TRANSFER FUNCTION**
Definitions:

Roots of numerator: ZEROS

Roots of denominator: POLES

\( m \leq n \): Proper Transfer Function

\( m < n \): Strictly Proper Transfer Function
Example – 1:
Simple First Order System (R-L Circuit)

\[ v(t) = L \frac{di(t)}{dt} + R i(t) \]

laplace transform

\[ \frac{I(s)}{V(s)} = \frac{1}{Ls + R} \quad \text{pole} = -\frac{R}{L} \]
Example – 2: (Second-order system) Transfer Function Modeling of Car Suspension System

\[ X_o = \text{Vertical motion of the body} \]

\[ X_i = \text{motion of the body at the point P} \]
Car Suspension System

\[ m\ddot{x}_0 + b(\dot{x}_0 - \dot{x}_i) + k(x_0 - x_i) = 0 \]

\[ m\ddot{x}_0 + b\dot{x}_0 + kx_0 = b\dot{x}_i + kx_i \]

Taking Laplace Transform

\[ (ms^2 + bs + k)X_0(s) = (bs + k)X_i(s) \]

Hence

\[ T(s) = \frac{X_0(s)}{X_i(s)} = \frac{(bs + k)}{(ms^2 + bs + k)} \]
Response of First and Second Order Systems

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**System Response: R-L Circuit**

\[ v(t) = L \frac{di(t)}{dt} + R i(t) \]

Laplace transform

\[ \frac{I(s)}{V(s)} = \frac{1}{Ls + R} \]

Pole location: \(-\frac{R}{L}\)

Let \( L = 1H \), \( R = 1\Omega \) and \( v(t) = 1V \) (unit step)
System Response: R-L Circuit

\[ \frac{I(s)}{V(s)} = \frac{1}{s + 2}; \quad \text{pole} = -2 \]

\[ V(s) = \frac{1}{s}; \quad \text{pole} = 0 \]

\[ I(s) = \frac{1}{s(s + 2)} \]

Partial fraction expansion

\[ I(s) = \frac{A}{s} + \frac{B}{s + 2} \]

\[ A = \left. \frac{1}{s + 2} \right|_{s \to 0} \]

\[ B = \left. \frac{1}{s} \right|_{s \to -2} \]

Taking Inverse Laplace Transform

\[ i(t) = \frac{1}{2} - \frac{1}{2}e^{-2t} \]

forced response  \quad  natural response
System Response: R-L Circuit

\[ \text{Total response} = \text{Forced response} + \text{Natural response} \]

- A Pole of the input function generates the form of the forced response.
- A Pole of the system transfer function generates the form of the natural response.
- The zeros and poles together generate the exact amplitudes for both forced and natural responses.
System Response: R-L Circuit

A system is stable if the natural response approaches zero as time approaches infinity. This demands $e^{-\alpha t}$ form in the natural response that means all the poles should lie in the left half of the s-plane.
First Order Systems

\[ \tau = \left( \frac{1}{a} \right) : \text{Time constant of the system} \]
Unit Step Response of First-Order System

Output response for a unit step input

\[ c(t) = 1 - e^{-t/\tau}, \quad \text{for} \quad t \geq 0 \]

The output will reach its final value as \( t \to \infty \).

Initial speed of response:

\[
\frac{dc}{dt} = \left( \frac{e^{-t/\tau}}{\tau} \right) \bigg|_{t=0} = \frac{1}{\tau}
\]
Unit Step Response of a First-Order System

Settling time: $T_s = 4T = 4\tau$
Second-Order System (R-L-C Circuit)

- $\zeta =$ damping ratio
- $\omega_n =$ un damped natural frequency
- Complex poles

\[
\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_ns + \omega_n^2}
\]
Unit Step Response
Second-Order System

\[ c(t) = 1 - \frac{e^{-\zeta \omega_n t}}{\sqrt{1 - \zeta^2}} \cos(\omega_d t - \phi), \quad \text{where } \omega_d = \omega_n \sqrt{1 - \zeta^2}, \quad \phi = \tan^{-1} \left( \frac{\zeta}{\sqrt{1 - \zeta^2}} \right) \]
Transient Response Specifications

For $t > t_p$, response remains within this strip.

These points are specified.

$M_p$, $t_d$, $t_r$, $t_p$, $t_s$.
Transient response specifications of an Under-damped system

Rise time \( T_r = \frac{\pi - \beta}{\omega_d} \),  
Peak time \( T_p = \frac{\pi}{\omega_d} \)

where \( \beta = \tan^{-1}\left(\frac{\omega_d}{\xi \omega_n}\right) \),  
\( \omega_d = \omega_n \sqrt{1 - \zeta^2} \)

Maximum overshoot \( M_p = e^{-\pi \xi / \sqrt{1 - \zeta^2}} \)

Settling time \( T_s = \frac{4}{\xi \omega_n} \) (**2% criterion**)  
\[ = \frac{3}{\xi \omega_n} \] (**5% criterion**)
Second-Order Systems: Pole Locations and Step Responses

- Over damped
- Under damped
- Undamped
- Critically damped
Second-order Response As A Function Of Damping Ratio

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<th>$\zeta$</th>
<th>Poles</th>
<th>Step response</th>
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<tr>
<td>0</td>
<td>$j\omega$, $-j\omega$</td>
<td>Undamped</td>
</tr>
<tr>
<td>$0 &lt; \zeta &lt; 1$</td>
<td>$-\zeta \omega_n$, $j\omega \sqrt{1 - \zeta^2}$</td>
<td>Underdamped</td>
</tr>
<tr>
<td>$\zeta = 1$</td>
<td>$-\zeta \omega_n$, $-\zeta \omega_n$</td>
<td>Critically damped</td>
</tr>
<tr>
<td>$\zeta &gt; 1$</td>
<td>$-\zeta \omega_n - \omega_n \sqrt{\zeta^2 - 1}$</td>
<td>Overdamped</td>
</tr>
</tbody>
</table>
Under Damped System Pole Plot

\[ +j \omega_n \sqrt{1 - \zeta^2} = j \omega_d \]

\[ -j \omega_n \sqrt{1 - \zeta^2} = -j \omega_d \]
Step Responses of Second Order Under Damped Systems with Pole Movement

- Envelop same
- Frequency same
- Same Overshoot

with constant real part

with constant imaginary part

with constant damping ratio
Effect of Adding a Pole

non dominant pole is near dominant second-order pair

far from the pair

At Infinity

Residue of non dominant pole and its response becomes zero as the non dominant pole approaches infinity
Effect of Adding a Zero

- Zeros and Poles together dictate the exact response (including magnitude)
- Zeros mainly effect the residues (i.e. the constants in the numerator in the partial fraction expansion)
- Closer the zero is to the dominant poles, the greater is its effect on the transient response
**Effect of Adding a Zero: Analysis**

Let \( C(s) \): Response of a system with unity in the numerator. Then by adding a zero, the Laplace transform of the response of the new system will be 

\[
(a + s)C(s) = aC(s) + sC(s)
\]

\( aC(s) \): A scaled version of the original response

\( sC(s) \): The derivative of the original response

Thus, if \( a \) is small (in the LH plane), the derivative term is predominant. Hence, more overshooting is expected.
Effect of Adding a Zero for Small Values of \( a \) in the Left-half \( s \)-plane

For small values of ‘\( a \)’, one gets more overshooting

![Graph showing the effect of adding a zero for small values of \( a \) in the Left-half \( s \)-plane.](image-url)
Effect of Adding a Zero in RHS of s-plane

\[(a - s)C(s) = aC(s) - sC(s), \quad a > 0\]

In this case the scaled response and derivative terms oppose each other!

Thus, if the derivative term is large, then the system response will initially follow the derivative "in the opposite direction" of the scaled response!
Effect of Adding a Zero in the Right Half s-plane

The system is called non-minimum phase

*Note:* Tail-controlled aerospace vehicles are typical examples for non-minimum phase systems
Thanks for the Attention...!