Lecture – 18

Time Response of Linear Dynamical Systems in State Space Form

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Solution of Linear Differential Equations

Linear systems:
Systems that obey the “Principle of superposition”.

Uniqueness Theorem:
There is only one solution for linear systems.
Solution of Homogeneous Linear Differential Equation: Scalar case

System dynamics: \[ \dot{x} = ax, \quad x(t_0) = x_0 \]

Solution:
\[ (dx / x) = a \, dt \]

\[ \ln x = a \, t + \ln c \]

\[ \ln \left( \frac{x}{c} \right) = a \, t \]

\[ x = e^{at} \, c \]

Initial condition:
\[ x_0 = e^{a t_0} \, c, \quad c = e^{-a t_0} \, x_0 \]

Hence,
\[ x(t) = e^{a(t-t_0)} \, x_0 \]
Solution of Homogeneous Linear Differential Equation: Scalar case

Note:

\[ e^{at} = 1 + at + \frac{a^2 t^2}{2!} + \frac{a^3 t^3}{3!} + \cdots \]

If \( t_0 = 0 \), then the solution is

\[ x(t) = e^{at} x_0 \]
Solution of Homogeneous Linear Differential Equations

System dynamics: \[ \dot{X} = AX, \quad X(t_0) = X_0 \]

Guess solution: \[ X(t) = e^{At} C, \quad C = [c_1 \ldots c_n]^T \]

\[ e^{At} \triangleq I + At + A^2 t^2 / 2! + A^3 t^3 / 3! + \ldots \]

Verify (substitute the guess into the differential equation)

\[ \left( \frac{d}{dt} e^{At} \right) C = A \left( e^{At} C \right) \]
Solution of Homogeneous Linear Differential Equations

A Result: \[ e^{At} = I + At + A^2 t^2 / 2! + A^3 t^3 / 3! + \cdots \]

\[
\frac{d}{dt}(e^{At}) = 0 + A + A^2 \left(2t / 2!\right) + A^3 \left(3t^2 / 3!\right) + \cdots \\
= A \left(I + At + A^2 t^2 / 2! + A^3 t^3 / 3! + \cdots\right) \\
= Ae^{At}
\]

i.e. \( (Ae^{At}) C = A(e^{At} C) \)

Therefore \( X(t) = e^{At} C \) is 'a' solution.

Hence, \( X(t) = e^{At} C \) is 'the' solution.
Solution of Homogeneous Linear Differential Equations

- Applying the initial condition
  \[ X_0 = e^{At_0} C \]
  \[ C = \left[ e^{At_0} \right]^{-1} X_0 \]

- Another result:
  \[ e^{A(t_1 + t_2)} = e^{At_1} e^{At_2} \]
  (easy to show from definition)

Taking \( t_1 = t_0 \) and \( t_2 = -t_0 \), \( I = e^{At_0} e^{-At_0} \)

Thus
\[ \left[ e^{At_0} \right]^{-1} = e^{-At_0} \]

Finally
\[ X(t) = e^{At} e^{-At_0} X_0 = e^{A(t-t_0)} X_0 \]
Solution of Non-homogeneous Linear Differential Equations

Non-homogeneous system: \( \dot{X} = AX + BU , \ X(t_0) = X_0 \)

Solution contains two parts:

- Homogeneous solution
- Particular solution

Homogeneous solution: \( X_h(t) = e^{A(t-t_0)}X_0 \)

Particular solution: \( X_p(t) = e^{At}C(t) \)
Solution of Non-homogeneous Linear Differential Equations

\[ \dot{X}_p = e^{At} \dot{C} + A e^{At} C = A e^{At} C + BU \]

\[ \dot{C} = e^{-At} BU \]

\[ C(t) = \int_{t_1}^{t} e^{-A\tau} BU(\tau) d\tau \]

\[ X_p(t) = e^{At} C(t) = e^{At} \int_{t_1}^{t} e^{-A\tau} BU(\tau) d\tau \]

\[ \quad = \int_{t_1}^{t} e^{A(t-\tau)} BU(\tau)d\tau \]
Solution of Non-homogeneous Linear Differential Equations

Complete solution: \[ X(t) = X_h(t) + X_p(t) \]

\[ = e^{A(t-t_0)}X_0 + \int_{t_1}^{t} e^{A(t-\tau)} BU(\tau) d\tau \]

Initial condition: At \( t = t_0 \)

\[ X_0 = X_0 + \int_{t_1}^{t_0} e^{A(t-\tau)} BU(\tau) d\tau \]

\[ \int_{t_1}^{t_0} e^{A(t-\tau)} BU(\tau) d\tau = 0 \]

This suggests that \( t_1 = t_0 \)
Solution of Non-homogeneous Linear Differential Equations

Complete solution:

\[ X(t) = e^{A(t-t_0)}X_0 + \int_{t_0}^{t} e^{A(t-\tau)}BU(\tau)d\tau \]

The integral term in the forced system solution is a convolution integral.

**Note:** If \( U \) is in feedback form \( (U = -KX) \)

\[ \dot{X} = (A - BK)X = A_{CL}X \]

\[ X(t) = e^{A_{CL}(t-t_0)}X_0 \]
The solution results do not demand that \( t \geq t_0 \). They are equally valid even if \( t \leq t_0 \).

The integral term in the forced system solution is a "convolution integral". i.e. The contribution of input \( U(t) \) is the convolution of \( U(t) \) with \( e^{At} B \).

Hence, the function \( e^{At} B \) has the role of "impulse response" of the system whose output is \( X(t) \) and input is \( U(t) \).

The solution for output \( Y(t) \) is also readily available from \( X(t) \) and \( U(t) \):

\[
Y(t) = CX(t) + DU(t)
\]
Example: Motion of a car without friction

The equation of motion is

\[ m\ddot{x} = f(t) \]
\[ \ddot{x} = \left(\frac{1}{m}\right) f(t) \quad \text{Assumption: } m \text{ is constant.} \]

\[ \nu \triangleq \dot{x} \]

\[ \dot{X} = \begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} f(t), \quad X(0) = \begin{bmatrix} x_0 \\ \nu_0 \end{bmatrix} \]
Example: Motion of a car without friction

\[ X(t) = e^{At} X_0 + \int_0^t e^{A(t-\tau)} B f(\tau) \, d\tau \]

\[ e^{At} = I + At + A^2 \frac{t^2}{2!} + \cdots = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ t \end{bmatrix} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \]

\[ e^{A(t-\tau)} = \begin{bmatrix} 1 & t - \tau \\ 0 & 1 \end{bmatrix} \]

\[ e^{A(t-\tau)} B = \begin{bmatrix} 1 & t - \tau \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1/m \end{bmatrix} = \frac{1}{m} \begin{bmatrix} t - \tau \\ 1 \end{bmatrix} \]
Example: Motion of a car without friction

\[
\begin{bmatrix}
x(t) \\
v(t)
\end{bmatrix} = \begin{bmatrix}
x_0 + v_0 t + \frac{1}{m} \int_0^t (t - \tau) f(\tau) d\tau \\
v_0 + \frac{1}{m} \int_0^t f(\tau) d\tau
\end{bmatrix} = \begin{bmatrix}
x_0 + v(t)t - \frac{1}{m} \int_0^t f(\tau) d\tau \\
v_0 + \frac{1}{m} \int_0^t f(\tau) d\tau
\end{bmatrix}
\]

Special case: \( f(t) / m = a \) (constant) and

\[
\begin{bmatrix}
x_0 \\
v_0
\end{bmatrix} = \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

\[
\begin{bmatrix}
x(t) \\
v(t)
\end{bmatrix} = \begin{bmatrix}
x_0 + (v_0 + at)t - \frac{t^2}{2} a \\
v_0 + at
\end{bmatrix} = \begin{bmatrix}
\frac{1}{2} at^2 \\
at
\end{bmatrix}
\]
Evaluation of $e^{At}$: A Useful Result

Problem: $\dot{X} = AX, \quad X(0) = X_0$

Solution using Laplace transform:

$sX(s) - X_0 = AX(s)$

$(sI - A)X(s) = X_0$

$X(s) = (sI - A)^{-1} X_0$

$X(t) = L^{-1} \left[ (sI - A)^{-1} \right] X_0$

Solution known:

$X(t) = e^{At} X_0$

Comparing the two solutions:

$e^{At} = L^{-1} \left[ (sI - A)^{-1} \right]$
Evaluation of $e^{At}$: How to compute it symbolically?

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \quad (sI - A) = \begin{bmatrix} s - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & s - a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & s - a_{nn} \end{bmatrix}$$

$$(sI - A)^{-1} = \frac{\text{adj}(sI - A)}{|sI - A|}$$

$$|sI - A| = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_n$$

$$\text{adj}(sI - A) = E_1 s^{n-1} + E_2 s^{n-2} + \cdots + E_n$$
Symbolic computation of $e^{At}$

\[
(sI - A)(sI - A)^{-1} = \frac{(sI - A)(E_1 s^{n-1} + E_2 s^{n-2} + \cdots + E_n)}{s^n + a_1 s^{n-1} + a_2 s^{n-1} + \cdots + a_n}
\]

\[
I\left(s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_n\right) = (sI - A)(E_1 s^{n-1} + E_2 s^{n-2} + \cdots + E_n)
\]

\[
s^n I + a_1 s^{n-1} I + a_2 s^{n-2} I + \cdots + a_n I
\]

\[
= s^n E_1 + s^{n-1} \left(E_2 - AE_1\right) + \cdots + s \left(E_n - AE_{n-1}\right) - AE_n
\]

Equate the coefficients on both sides…
Symbolic computation of $e^{At}$

\[
E_1 = I \\
E_2 - AE_1 = a_1 I \\
E_3 - AE_2 = a_2 I \\
\vdots \\
E_n - AE_{n-1} = a_{n-1} I \\
- AE_n = a_n I
\]

This suggests a recursive algorithm! (provided $a_1, \ldots, a_n$ are known)

\[
a_i = -\left(\frac{1}{i}\right) Tr\left(AE_i\right)
\]

for $i = 1, \ldots, n$

Solution of Linear Time Varying Systems

Homogeneous Linear System \( \dot{X} = A(t)X \)

Solution: \( X(t) = \varphi(t, \tau)X(\tau) \)

\( \varphi(t, \tau) \): State Transition Matrix (STM)

PROPERTIES OF STM

1. It satisfies linear differential equation \( \frac{\partial \varphi}{\partial t} = A(t)\varphi(t, \tau) \)

2. \( \varphi(t, t) = I \)

3. For any three time instants \( \varphi(t_3, t_1) = \varphi(t_3, t_2)\varphi(t_2, t_1) \)
Properties of STM

4. $\varphi(\tau, t) = \left[ \varphi(t, \tau) \right]^{-1}$

5. For time-invariant systems
   $\varphi(0) = I$
   $\varphi(t)\varphi(\tau) = \varphi(t + \tau)$
   $\varphi^{-1}(t) = \varphi(-t)$

6. For linear time invariant system
   $\varphi(t, \tau) = e^{A(t-\tau)}$
Solution of Linear Time Varying Systems

**Solution:** \( X(t) = \varphi(t, t_0) C(t) \)  
(Method of variation of parameters)

How to determine \( C(t) \)?

\[
\dot{X} = AX + BU \\
\left[ \frac{\partial \varphi}{\partial t} C \right] + \varphi \dot{C} = \left[ A \varphi C \right] + BU \quad , \quad \dot{C} = \left[ \varphi(t, t_0) \right]^{-1} BU
\]

\[
C(t) = C(t_0) + \int_{t_0}^{t} \left[ \varphi^{-1}(\tau, t_0) \right] B(\tau) U(\tau) \, d\tau
\]

\[
X(t_0) = C(t_0), \quad \varphi(t_0, t_0) = I
\]
Solution of Linear Time Varying Systems

\[
X(t) = \varphi(t,t_0) \left[ X(t_0) + \int_{t_0}^{t} \varphi^{-1}(\tau,t_0) B(\tau) U(\tau) \, d\tau \right]
\]

\[
= \varphi(t,t_0) X(t_0) + \int_{t_0}^{t} \left[ \varphi(t,t_0) \varphi(t_0,\tau) \right] B(\tau) U(\tau) \, d\tau
\]

\[
X(t) = \varphi(t,t_0) X(t_0) + \int_{t_0}^{t} \varphi(t,\tau) B(\tau) U(\tau) \, d\tau
\]
Thanks for the Attention...!