Introduction to
Machine-Independent Optimizations - 3
Data-Flow Analysis

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NPTEL Course on Principles of Compiler Design
Outline of the Lecture

- What is code optimization? (in part 1)
- Illustrations of code optimizations (in part 1)
- Examples of data-flow analysis
- Fundamentals of control-flow analysis
- Algorithms for two machine-independent optimizations
- SSA form and optimizations
Sets of expressions constitute the domain of data-flow values.

Forward flow problem

Confluence operator is $\cap$

An expression $x + y$ is available at a point $p$, if every path (not necessarily cycle-free) from the initial node to $p$ evaluates $x + y$, and after the last such evaluation, prior to reaching $p$, there are no subsequent assignments to $x$ or $y$.

A block kills $x + y$, if it assigns (or may assign) to $x$ or $y$ and does not subsequently recompute $x + y$.

A block generates $x + y$, if it definitely evaluates $x + y$, and does not subsequently redefine $x$ or $y$. 
In other blocks:

\[
\begin{align*}
d5: b &= a + 4 \\
d6: f &= e + c \\
d7: e &= b + d \\
d8: d &= a + b \\
d9: a &= c + f \\
d10: c &= e + a
\end{align*}
\]

\[
\begin{align*}
d1: a &= f + 1 \\
d2: b &= a + 7 \\
d3: c &= b + d \\
d4: a &= d + c
\end{align*}
\]

Set of all expressions = \{f+1,a+7,b+d,d+c,a+4,e+c,a+b,c+f,e+a\}

\[
\begin{align*}
\text{EGEN}[B] &= \{f+1,b+d,d+c\} \\
\text{EKILL}[B] &= \{a+4,a+b,e+a,e+c,c+f,a+7\}
\end{align*}
\]
The data-flow equations

\[
\begin{align*}
  \text{IN}[B] &= \bigcap_{P \text{ is a predecessor of } B} \text{OUT}[P], \text{ B not initial} \\
  \text{OUT}[B] &= \text{e}_\text{gen}[B] \bigcup (\text{IN}[B] - \text{e}_\text{kill}[B]) \\
  \text{IN}[B_1] &= \phi \\
  \text{IN}[B] &= U, \text{ for all } B \neq B_1 \text{ (initialization only)}
\end{align*}
\]

- $B_1$ is the initial or entry block and is special because nothing is available when the program begins execution.
- $\text{IN}[B_1]$ is always $\phi$.
- $U$ is the universal set of all expressions.
- Initializing $\text{IN}[B]$ to $\phi$ for all $B \neq B_1$, is restrictive.
Available Expression Computation - DF Equations (2)

\[ \text{IN}[B] = \bigcup_{P \text{ is a predecessor of } B} \text{OUT}[P], \text{ B not initial} \]
\[ \text{OUT}[B] = e_{\text{gen}}[B] \cup (\text{IN}[B] - e_{\text{kill}}[B]) \]

\[ \text{IN}[B4] = \text{OUT}[B1] \cap \text{OUT}[B2] \cap \text{OUT}[B3] \]
\[ \text{OUT}[B4] = e_{\text{gen}}[B4] \cup (\text{IN}[B4] - e_{\text{kill}}[B4]) \]

\[ \text{IN}[B4] \{a+b,p+q\} \rightarrow B4 \]
\[ \text{e}_{\text{gen}}[B4] \{x+y\} \rightarrow B4 \]
\[ \text{e}_{\text{kill}}[B4] \{a+b\} \rightarrow B4 \]

\[ \text{OUT}[B4] \{x+y,p+q\} \]
Available Expression Computation - An Example (2)

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Data-Flow Analysis
An Iterative Algorithm for Computing Available Expressions

for each block $B \neq B_1$ do { $OUT[B] = U - e_{kill}[B]$; }

/* You could also do $IN[B] = U$; */

/* In such a case, you must also interchange the order of */

/* $IN[B]$ and $OUT[B]$ equations below */

$change = true$;

while $change$ do { $change = false$;
    for each block $B \neq B_1$ do {

        $IN[B] = \bigcap_{P \text{ a predecessor of } B} OUT[P]$;

        $oldout = OUT[B]$;

        $OUT[B] = e_{gen}[B] \cup (IN[B] - e_{kill}[B])$;

        if ($OUT[B] \neq oldout$) $change = true$;
    }
}

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Live Variable Analysis

- The variable $x$ is *live* at the point $p$, if the value of $x$ at $p$ could be used along some path in the flow graph, starting at $p$; otherwise, $x$ is *dead* at $p$.
- Sets of variables constitute the domain of data-flow values.
- Backward flow problem, with confluence operator $\bigcup$.
- $IN[B]$ is the set of variables live at the beginning of $B$.
- $OUT[B]$ is the set of variables live just after $B$.
- $DEF[B]$ is the set of variables definitely assigned values in $B$, prior to any use of that variable in $B$.
- $USE[B]$ is the set of variables whose values may be used in $B$ prior to any definition of the variable.

$$
OUT[B] = \bigcup_{S \text{ is a successor of } B} IN[S]
$$

$$
IN[B] = USE[B] \bigcup (OUT[B] - DEF[B])
$$

$$
IN[B] = \emptyset, \text{for all } B \text{ (initialization only)}
$$
Live Variable Analysis: An Example - Pass 1

Pass 1

entry

i := m-1
j := n
a := u1

USE[B1] = \{m, n, u1\}
DEF[B1] = \{i, j, a\}
IN[B1] = \{m, n, u1\}
OUT[B1] = \{\}

USE[B2] = \{i, j\}
DEF[B2] = \{\}
IN[B2] = \{i, j\}
OUT[B2] = \{\}

i := i+1
j := j-1

USE[B3] = \{u2\}
DEF[B3] = \{a\}
IN[B3] = \{u2\}
OUT[B3] = \{\}

a := u2

USE[B4] = \{a, j\}
DEF[B4] = \{i\}
IN[B4] = \{a, j\}
OUT[B4] = \{\}

i := a+j

exit

\[ \text{OUT}[B] = \bigcup_{S \text{ is a successor of } B} \text{IN}[S] \]

\[ \text{IN}[B] = \text{USE}[B] \bigcup (\text{OUT}[B] - \text{DEF}[B]) \]
Live Variable Analysis: An Example - Pass 2.1

Pass 2

**entry**

**B1**
- USE[B1] = \{m, n, u1\}
- DEF[B1] = \{i, j, a\}
- IN[B1] = \{m, n, u1\}
- OUT[B1] = \{i, j\}

\[i := m-1\]
\[j := n\]
\[a := u1\]

**B2**
- USE[B2] = \{i, j\}
- DEF[B2] = \{
- IN[B2] = \{i, j\}
- OUT[B2] = \{

\[i := i+1\]
\[j := j-1\]

**B3**
- USE[B3] = \{
- DEF[B3] = \{
- IN[B3] = \{a\}
- OUT[B3] = \{

\[a := u2\]

**B4**

\[i := a+j\]

**exit**

\[OUT[B] = \bigcup S \text{ is a successor of } B\]
\[IN[B] = USE[B] \bigcup (OUT[B] - DEF[B])\]
Live Variable Analysis: An Example - Pass 2.2

Pass 2

entry

B1

i := m-1
j := n
a := u1

DEF[B1]={i,j,a}
IN[B1]={m,n,u1}
OUT[B1]={i,j}

USE[B1]={m,n,u1}

B2

i := i+1
j := j-1

DEF[B2]={}
IN[B2]={i,j,u2,a}
OUT[B2]={a,j,u2}

USE[B2]={i,j}

a := u2

B3

USE[B3]={u2}
DEF[B3]={a}
IN[B3]={u2}
OUT[B3]={}

B4

USE[B4]={a,j}
DEF[B4]={i}
IN[B4]={a,j}
OUT[B4]={}

B2

exit

\[ \text{OUT}[B] = \bigcup \text{IN}[S] \]
\[ \text{S is a successor of } B \]
\[ \text{IN}[B] = \text{USE}[B] \cup (\text{OUT}[B] - \text{DEF}[B]) \]
Live Variable Analysis: An Example - Pass 2.3

Pass 2

entry

B1

\[ i := m-1 \]
\[ j := n \]
\[ a := u1 \]

\( \text{USE}[B1]=\{m,n,u1\} \)
\( \text{DEF}[B1]=\{i,j,a\} \)
\( \text{IN}[B1]=\{m,n,u1\} \)
\( \text{OUT}[B1]=\{i,j\} \)

B2

\[ i := i+1 \]
\[ j := j-1 \]

\( \text{USE}[B2]=\{i,j\} \)
\( \text{DEF}[B2]=\{} \)
\( \text{IN}[B2]=\{i,j,u2,a\} \)
\( \text{OUT}[B2]=\{a,j,u2\} \)

B3

\[ a := u2 \]

\( \text{USE}[B3]=\{u2\} \)
\( \text{DEF}[B3]=\{a\} \)
\( \text{IN}[B3]=\{j,u2\} \)
\( \text{OUT}[B3]=\{a,j\} \)

B4

\[ i := a+j \]

\( \text{USE}[B4]=\{a,j\} \)
\( \text{DEF}[B4]=\{i\} \)
\( \text{IN}[B4]=\{a,j\} \)
\( \text{OUT}[B4]=\{} \)

exit

\[ \text{OUT}[B] = \bigcup S \text{ is a successor of } B \]
\[ \text{IN}[B] = \text{USE}[B] \bigcup (\text{OUT}[B] - \text{DEF}[B]) \]
**Live Variable Analysis: An Example - Pass 2.4**

**Pass 2**

**B1**
- \( i := m-1 \)
- \( j := n \)
- \( a := u1 \)

**USE[B1]=\{m,n,u1\}**
**DEF[B1]=\{i,j,a\}**
**IN[B1]=\{m,n,u1\}**
**OUT[B1]=\{i,j\}**

**B2**
- \( i := i+1 \)
- \( j := j-1 \)

**USE[B2]=\{i,j\}**
**DEF[B2]=\{\}**
**IN[B2]=\{i,j,u2,a\}**
**OUT[B2]=\{a,j,u2\}**

**B3**
- \( a := u2 \)

**USE[B3]=\{u2\}**
**DEF[B3]=\{a\}**
**IN[B3]=\{j,u2\}**
**OUT[B3]=\{a,j\}**

**B4**
- \( i := a+j \)

**USE[B4]=\{a,j\}**
**DEF[B4]=\{i\}**
**IN[B4]=\{a,j,u2\}**
**OUT[B4]=\{a,i,j,u2\}**

**exit**

**\( \text{OUT} [B] = \bigcup S \text{ is a successor of } B \)**

**\( \text{IN} [B] = \text{USE} [B] \cup (\text{OUT} [B] - \text{DEF} [B]) \)**
Live Variable Analysis: An Example - Final pass

Pass 3 (final)

entry

i := m-1
j := n
a := u1

B1

USE[B1] = {m, n, u1}
DEF[B1] = {i, j, a}
IN[B1] = {m, n, u1, u2}
OUT[B1] = {i, j, u2, a}

USE[B2] = {i, j}
DEF[B2] = {}
IN[B2] = {i, j, u2, a}
OUT[B2] = {a, j, u2}

i := i+1
j := j-1

B2

USE[B3] = {u2}
DEF[B3] = {a}
IN[B3] = {j, u2}
OUT[B3] = {a, j, u2}

a := u2

B3

USE[B4] = {a, j}
DEF[B4] = {i}
IN[B4] = {a, j, u2}
OUT[B4] = {a, i, j, u2}

i := a+j

B4

exit

OUT[B] = \bigcup S \text{ is a successor of } B
IN[B] = USE[B] \bigcup (OUT[B] - DEF[B])

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Data-Flow Analysis
Data-flow Analysis: Theoretical Foundations

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Basic questions to be answered

1. In which situations is the iterative DFA algorithm correct?
2. How precise is the solution produced by it?
3. Will the algorithm converge?
4. What is the meaning of a “solution”?

The above questions can be answered accurately by a DFA framework.

Further, reusable components of the DFA algorithm can be identified once a framework is defined.

A DFA framework \((D, V, \wedge, F)\) consists of:
- \(D\) : A direction of the dataflow, either forward or backward
- \(V\) : A domain of values
- \(\wedge\) : A meet operator; \((V, \wedge)\) form a semi-lattice
- \(F\) : A family of transfer functions, \(V \rightarrow V\)

\(F\) includes constant transfer functions for the ENTRY/EXIT nodes as well.
A semi-lattice is a set $V$ and a binary operator $\land$, such that the following properties hold:

1. $V$ is closed under $\land$.
2. $\land$ is idempotent ($x \land x = x$), commutative ($x \land y = y \land x$), and associative ($x \land (y \land z) = (x \land y) \land z$).
3. It has a top element, $\top$, such that $\forall x \in V$, $\top \land x = x$.
4. It may have a bottom element, $\bot$, such that $\forall x \in V$, $\bot \land x = \bot$.

The operator $\land$ defines a partial order $\leq$ on $V$, such that $x \leq y$ iff $x \land y = x$. 
3 definitions, \{d1,d2,d3\}

\( V \) is the set of all subsets of \{d1,d2,d3\}

\( \wedge \) is \( \cup \)

The diagram (next slide) shows the partial order relation induced by \( \wedge \) (i.e., \( \cup \))

Partial order relation is \( \supseteq \)

An arrow, \( y \rightarrow x \) indicates \( x \supseteq y \) \( (x \leq y) \)

Each set in the diagram is a data-flow value

Transitivity is implied in the diagram \( (a \rightarrow b \& b \rightarrow c \implies a \rightarrow c) \)

An ascending chain: \( (x_1 < x_2 < \ldots < x_n) \)

Height of a semi-lattice: largest number of ‘<’ relations in any ascending chain

Semi-lattices in our DF frameworks will be of finite height
$y \to x$ indicates $x \supseteq y$ ($x \leq y$)
Transfer Functions

\[ F : V \rightarrow V \] has the following properties

1. \( F \) has an identity function, \( I(x) = x \), for all \( x \in V \)
2. \( F \) is closed under composition, \( i.e., \) for \( f, g \in F \), \( f \cdot g \in F \)

**Example:** Again considering the R-D problem

- Assume that each quadruple is in a separate basic block
- \( OUT[B] = GEN[B] \cup (IN[B] - KILL[B]) \)
- In its general form, this becomes \( f(x) = G \cup (x - K) \)
- \( F \) consists of such functions \( f \), one for each basic block
- Identity function exists here (when both \( G \) and \( K \) (\( GEN \) and \( KILL \)) are empty)
Transfer functions:

\[ f_{d1}(x) = \{d1\} \cup (x - \{d4\}) \]
\[ f_{d2}(x) = \{d2\} \cup (x - \{d3\}) \]
\[ f_{d3}(x) = \{d3\} \cup (x - \{d2\}) \]
\[ f_{d4}(x) = \{d4\} \cup (x - \{d1\}) \]
\[ f_{d5}(x) = \{d5\} \cup (x - \Phi) \]

Transfer functions for start and stop blocks are identity functions

\[ f_{B1} = (f_{d2} \cdot f_{d1})(x) \]
\[ = \{d2\} \cup (\{d1\} \cup (x - \{d4\}) - \{d3\}) \]
\[ = \{d1, d2\} \cup (x - \{d3, d4\}) \]
\[ f_{B2} = (f_{d4} \cdot f_{d3})(x) \]
\[ = \{d3, d4\} \cup (x - \{d1, d2\}) \]
\[ f_{B3} = f_{d5} = \{d5\} \cup x \]
A DF framework \((D, F, V, \wedge)\) is monotone, if
\[\forall x, y \in V, f \in F, x \leq y \Rightarrow f(x) \leq f(y), \text{ OR} \]
\[f(x \wedge y) \leq f(x) \wedge f(y)\]
The reaching definitions framework is monotone

A DF framework is distributive, if
\[\forall x, y \in V, f \in F, f(x \wedge y) = f(x) \wedge f(y)\]
Distributivity \(\Rightarrow\) monotonicity, but not vice-versa
The reaching definitions lattice is distributive
\[
\begin{align*}
\{ \text{OUT}[B_1] &= v_{init}; \\
\text{for each block } B \neq B_1 \text{ do } \text{OUT}[B] &= \top; \\
\text{while (changes to any OUT occur) do } \\
&\text{for each block } B \neq B_1 \text{ do} \\
&\quad \text{IN}[B] = \bigwedge_{P \text{ a predecessor of } B} \text{OUT}[P]; \\
&\quad \text{OUT}[B] = f_B(\text{IN}[B]); \\
\end{align*}
\]
Reaching Definitions Framework - Example contd.

Start

B1

\[ d1: a = 2 \]
\[ d2: b = 2 \]

B2

\[ d3: b = 3 \]
\[ d4: a = 3 \]

B3

\[ d5: c = a + b \]

stop

\[ f_{B1} = \{d1,d2\} \cup (x - \{d3,d4\}) \]
\[ f_{B2} = \{d3,d4\} \cup (x - \{d1,d2\}) \]
\[ f_{B3} = \{d5\} \cup x \]

\[ \text{IN}[B] = \bigwedge_{P, a \text{ predecessor of } B} \text{OUT}[P] \]
\[ = \bigcup_{P, a \text{ predecessor of } B} \text{OUT}[P] \]

\[ \text{OUT}[B] = f_B(\text{IN}[B]) \]

Needs 2 iterations to converge
\[ \text{IN}[B1] = \text{IN}[B2] = \emptyset; \text{OUT}[B1] = \{d1,d2\}; \text{OUT}[B2] = \{d3,d4\} \]
\[ \text{IN}[B3] = \text{OUT}[B1] \cup \text{OUT}[B2] = \{d1,d2,d3,d4\} \]
\[ \text{OUT}[B3] = \{d5\} \cup \text{IN}[B3] = \{d1,d2,d3,d4,d5\} \]