Module 11: Introduction to Optimal Control
Lecture Note 3

1 Linear Quadratic Regulator

Consider a linear system modeled by

$$x(k+1) = Ax(k) + Bu(k), \quad x(k_0) = x_0$$

where $x(k) \in \mathbb{R}^n$ and $u(k) \in \mathbb{R}^m$. The pair $(A, B)$ is controllable.

The objective is to design a stabilizing linear state feedback controller $u(k) = -Kx(k)$ which will minimize the quadratic performance index, given by,

$$J = \sum_{k=0}^{\infty} (x^T(k)Qx(k) + u^T(k)Ru(k))$$

where, $Q = Q^T \geq 0$ and $R = R^T > 0$. Such a controller is denoted by $u^\ast$.

We first assume that a linear state feedback optimal controller exists such that the closed loop system

$$x(k+1) = (A - BK)x(k)$$

is asymptotically stable.

This assumption implies that there exists a Lyapunov function $V(x(k)) = x(k)^TPx(k)$ for the closed loop system, for which the forward difference

$$\Delta V(x(k)) = V(x(k+1)) - V(x(k))$$

is negative definite.

We will now use the theorem as discussed in the previous lecture which says if the controller $u^\ast$ is optimal, then

$$\min_u (\Delta V(x(k)) + x^T(k)Qx(k) + u^T(k)Ru(k)) = 0$$
Now, finding an optimal controller implies that we have to find an appropriate Lyapunov function which is then used to construct the optimal controller.

Let us first find the $u^*$ that minimizes the function

$$ f = f(u(k)) = \Delta V(x(k)) + x^T(k)Qx(k) + u^T(k)Ru(k) $$

If we substitute $\Delta V$ in the above expression, we get

$$ f(u(k)) = x^T(k+1)Px(k+1) - x^T(k)Px(k) + x^T(k)Qx(k) + u^T(k)Ru(k) $$

Taking derivative of the above function with respect to $u(k)$,

$$ \frac{\partial f(u(k))}{\partial u(k)} = 2(Ax(k) + Bu(k))^TP(Ax(k) + Bu(k)) - x^T(k)Px(k) + x^T(k)Qx(k) + u^T(k)Ru(k) $$

The matrix $B^TPB + R$ is positive definite since $R$ is positive definite, thus it is invertible. Hence,

$$ u^*(k) = -(B^TPB + R)^{-1}B^TPAx(k) = -Kx(k) $$

where $K = (B^TPB + R)^{-1}B^TPA$. Let us denote $B^TPB + R$ by $S$. Thus

$$ u^*(k) = -S^{-1}B^TPAx(k) $$

We will now check whether or not $u^*$ satisfies the second order sufficient condition for minimization. Since

$$ \frac{\partial^2 f(u(k))}{\partial u^2(k)} = \frac{\partial}{\partial u(k)}(2x^T(k)A^TPB + 2u^T(k)(B^TPB + R)) $$

$$ = 2(B^TPB + R) > 0 $$

$u^*$ satisfies the second order sufficient condition to minimize $f$. 

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The optimal controller can thus be constructed if an appropriate Lyapunov matrix $P$ is found. For that let us first find the closed loop system after introduction of the optimal controller.

$$ x(k + 1) = (A - BS^{-1}B^T PA)x(k) $$

Since the controller satisfies the hypothesis of the theorem, discussed in the previous lecture,

$$ x^T(k + 1)Px(k + 1) - x^T(k)Px(k) + x^T(k)Qx(k) + u^T(k)Ru^*(k) = 0 $$

Putting the expression of $u^*$ in the above equation,

$$ x^T(k)(A - BS^{-1}B^T PA)^T P(A - BS^{-1}B^T PA)x(k) - x^T(k)Px(k) + x^T(k)Qx(k) + x^T(k)AT PBS^{-1}RS^{-1}B^T PAX(k) $$

$$ = x^T(k)AT PAx(k) - x^T(k)AT PBS^{-1}B^T PAX(k) - x^T(k)AT PBS^{-1}B^T PAX(k) - x^T(k)Px(k) + x^T(k)Qx(k) + x^T(k)AT PBS^{-1}RS^{-1}B^T PAX(k) $$

$$ = x^T(k)AT PAx(k) - x^T(k)Px(k) + x^T(k)Qx(k) - 2x^T(k)AT PBS^{-1}B^T PAX(k) + x^T(k)AT PBS^{-1}(B^T PB + R)S^{-1}B^T PAX(k) $$

$$ = x^T(k)AT PAx(k) - x^T(k)Px(k) + x^T(k)Qx(k) - 2x^T(k)AT PBS^{-1}S^{-1}B^T PAX(k) + x^T(k)AT PBS^{-1}B^T PAX(k) $$

$$ = x^T(k)(AT PA - P + Q - AT PBS^{-1}B^T PA)x(k) = 0 $$

The above equation should hold for any value of $x(k)$. Thus

$$ AT PA - P + Q - AT PBS^{-1}B^T PA = 0 $$

which is the well known discrete Algebraic Riccati Equation (ARE). By solving this equation we can get $P$ to form the optimal regulator to minimize a given quadratic performance index.

**Example 1:** Consider the following linear system

$$ x(k + 1) = \begin{bmatrix} 0 & 1 \\ 0.5 & 0.8 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k), \quad x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} $$

$$ y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(k) $$

Design an optimal controller to minimize the following performance index.

$$ J = \sum_{k=0}^{\infty} (x_1^2 + x_1 x_2 + x_2^2 + 0.1 u^2) $$
Also, find the optimal cost.

**Solution:** The performance index \( J \) can be rewritten as

\[
J = \sum_{k=0}^{\infty} (x^T(k) \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix} x(k) + 0.1u^2)
\]

Thus, \( Q = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix} \) and \( R = 0.1 \).

Let us take \( P \) as

\[
P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}
\]

Then,

\[
A^T PA - P = \begin{bmatrix} 0.25p_3 - p_1 & 0.5p_2 + 0.4p_3 - p_2 \\ 0.5p_2 + 0.4p_3 - p_2 & p_1 + 1.6p_2 + 0.64p_3 - p_3 \end{bmatrix}
\]

\[
A^T PA - P + Q = \begin{bmatrix} 0.25p_3 - p_1 + 1 & 0.4p_3 - 0.5p_2 + 0.5 \\ 0.4p_3 - 0.5p_2 + 0.5 & p_1 + 1.6p_2 - 0.36p_3 + 1 \end{bmatrix}
\]

\[
A^T PB = \begin{bmatrix} 0.5p_3 \\ p_2 + 0.8p_3 \end{bmatrix}, \quad B^T PA = [0.5p_3 \ p_2 + 0.8p_3], \quad S = 0.1 + p_3
\]

\[
A^T PBS^{-1}B^T PA = = \frac{1}{0.1 + p_3} \begin{bmatrix} 0.5p_3 \\ p_2 + 0.8p_3 \end{bmatrix} \begin{bmatrix} 0.5p_3 \\ p_2 + 0.8p_3 \end{bmatrix}
\]

\[
= \frac{1}{0.1 + p_3} \begin{bmatrix} 0.25p_3^2 & 0.5p_2p_3 + 0.4p_3^2 \\ 0.5p_2p_3 + 0.4p_3^2 & p_2^2 + 1.6p_2p_3 + 0.64p_3^2 \end{bmatrix}
\]

The discrete ARE is

\[
A^T PA - P + Q - A^T PBS^{-1}B^T PA = 0
\]

Or,

\[
\begin{bmatrix}
0.25p_3 - p_1 + 1 - \frac{0.25p_3^2}{0.1 + p_3} & 0.4p_3 - 0.5p_2 + 0.5 - \frac{0.5p_2p_3 + 0.4p_3^2}{0.1 + p_3} \\
0.4p_3 - 0.5p_2 + 0.5 - \frac{0.5p_2p_3 + 0.4p_3^2}{0.1 + p_3} & p_1 + 1.6p_2 - 0.36p_3 + 1 - \frac{0.1 + p_3}{p_2^2 + 1.6p_2p_3 + 0.64p_3^2}
\end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
\]
We can get three equations from the discrete ARE. These are

\[ 0.25p_3 - p_1 + 1 - \frac{0.25p_3^2}{0.1 + p_3} = 0 \]

\[ 0.4p_3 - 0.5p + 0.5 - \frac{0.5p_2p_3 + 0.4p_3^2}{0.1 + p_3} = 0 \]

\[ p_1 + 1.6p_2 - 0.36p_3 + 1 - \frac{p_2^2 + 1.6p_2p_3 + 0.64p_3^2}{0.1 + p_3} = 0 \]

Since the above three equations comprises three unknown parameters, these parameters can be solved uniquely, as

\[ p_1 = 1.0238, \quad p_2 = 0.5513, \quad p_3 = 1.9811 \]

The optimal control law can be found out as

\[ u^*(k) = -(R + B^T PB)^{-1}B^T PAx(k) \]

\[ = -[0.4760 \ 1.0265]x(k) \]

\[ = -0.4760x_1(k) - 1.0265x_2(k) \]

The optimal cost can be found as

\[ J = x_0^TPx_0 \]

\[ = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1.0238 & 0.5513 \\ 0.5513 & 1.9811 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \]

\[ = 4.1075 \]