Proof of antisymmetry of the infinitesimal Lorentz parameters

A finite Lorentz transformation satisfies the equation

\[ \Lambda \cdot \eta \cdot \Lambda^T = \eta, \quad \Lambda^\mu_\nu \eta^{\nu \rho} \Lambda^\sigma_\rho = \eta^{\mu \sigma} \]  \hspace{1cm} (1)

**Remark:** Note that in the second equation above, the transposition is *implicitly* present in the second \( \Lambda \).

Expanding \( \Delta \) as follows where \( \lambda \) parametrises infinitesimal Lorentz transformations:

\[ \Lambda^\mu_\nu = \delta^\mu_\nu + \lambda^\mu_\nu + \mathcal{O}(\lambda^2). \]  \hspace{1cm} (2)

For instance, for a Lorentz boost with velocity \( \mathbf{u} = (u_1, u_2, u_3) \), one has (see [here](#))

\[ \lambda(\mathbf{u}) = \begin{pmatrix} 0 & -u_1/c & -u_2/c & -u_3/c \\ -u_1/c & 0 & 0 & 0 \\ -u_2/c & 0 & 0 & 0 \\ -u_3/c & 0 & 0 & 0 \end{pmatrix}. \]  \hspace{1cm} (3)

**Exercise:** Obtain the matrix \( \Lambda^\mu_\nu \) for the simple case of a boost in the \( x_1 \) direction i.e., with \( u_2 = u_3 = 0 \) by computing the matrix exponential. *Hint:* Compute \( \lambda^2 \) and see that it is proportional to the identity matrix.

On substituting Eq.(2) in Eq.(1), to first order in \( \lambda \), we get

\[ \lambda^\mu_\nu \eta^{\nu \rho} \delta^\sigma_\rho + \delta^\mu_\nu \eta^{\nu \rho} \lambda^\sigma_\rho = 0, \]  \hspace{1cm} (4)

\[ \Rightarrow \lambda^\mu_\nu \eta^{\nu \sigma} + \eta^{\mu \rho} \lambda^\sigma_\rho = 0, \]

\[ \Rightarrow \lambda^{\mu \sigma} + \lambda^{\sigma \mu} = 0. \]

We thus see that Eq.(1) implies that \( \lambda^{\mu \nu} := \lambda^\mu_\rho \eta^{\nu \rho} \) is *antisymmetric* as claimed in the lecture. In fact, one has the stronger statement:
\[ \Lambda^\mu_\nu = \exp(\lambda^\mu_\nu) . \] (5)