Lie Algebras

It is conventional to indicate the Lie algebra associated with a group $G$ by $g$, the corresponding letter in lower case. Thus the Lie algebra of the Lie group $SU(n)$ is written as $su(n)$ and so on.

1. A basis for the Lie Algebra of $so(N)$ is given by the following $N \times N$ matrices:

$$(M_{mn})_{ab} = \delta_{ma}\delta_{nb} - \delta_{mb}\delta_{na}.$$

Note that all indices $m, n, a, b$ run from 1 to $N$.

(a) Show that the commutator (Lie Bracket) of the basis matrices is given by

$$[M_{mn}, M_{pq}] = \delta_{np}M_{mq} - \delta_{mp}M_{nq} - \delta_{nq}M_{mp} + \delta_{mq}M_{nn}.$$

(b) Show that when $N = 3$, the (above) Lie algebra is isomorphic to the $su(2)$ Lie algebra.

(c) Show that when $N = 4$, the Lie algebra is isomorphic to $su(2) \oplus su(2)$.

$Hint$: Consider the two sets of linear combinations $(M_{ab} \pm \alpha\epsilon_{abc}M_{c4})$ for $a, b, c = 1, 2, 3$ for a suitable chosen $\alpha$.

(d) Show that dimensions of the Lie algebras $so(6)$ and $su(4)$ are the same. This is a necessary but not sufficient condition for $so(6) \sim su(4)$.

2. The $su(3)$ Lie algebra The $su(3)$ Lie algebra is generated by the LVS of $3 \times 3$ traceless hermitian matrices. A standard choice for this basis is given by the (eight) Gellmann $\lambda$-matrices:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$
(a) Verify that $\text{Tr}(\lambda_a \lambda_b) = 2\delta_{ab}$.

(b) The generators of $su(3)$ are chosen to be $T_a = \frac{1}{2}\lambda_a$. The Cartan subalgebra is chosen to be $H = (h_1 = T_3, h_2 = T_8)$. Let $\text{Ad}_{h_i}$ $(i = 1, 2)$ be the linear maps from the Lie algebra to itself defined as follows:

$$\text{Ad}_{h_i} : x \mapsto [h_i, x] \quad \forall x \in su(3).$$

Organize the remaining six generators such that they are simultaneous eigenvectors under the two maps i.e., $[h_i, x] = \alpha_i x$ for $i = 1, 2$.

(c) Hence, obtain the Cartan decomposition of $su(3) = L^+ \oplus H \oplus L^-$. Note that $L^\pm$ are both subalgebras of dimension three.

(d) The structure constants are defined by the following relation:

$$[T_a, T_b] = i f_{abc} \ T_c.$$

Obtain the structure constants and show that $f_{abc}$ is totally antisymmetric.

**Recommended Reading:** Equivalences of Lie Algebras:

http://sgovindarajan.wikidot.com/equivalence-lie-algebras