Optimal Control, Guidance and Estimation

Lecture – 37

Optimal Control of
Distributed Parameter Systems – I

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Distributed Parameter Systems (DPS)

Systems are Governed by a Set of
Partial Differential Equations

Examples:
- Heat Transfer Processes
- Fluid Flows
- Chemical Reactor Processes
- Vibration of Structures (Aeroelastic Problems)
- Ecological Problems
Control of Distributed Parameter Systems

- Design-then-Approximate
- Approximate-then-Design
  - Design without model reduction
  - Design with model reduction

Topics
- LQR
  - Using Finite Difference (through examples)
- Optimal Dynamic Inversion
  - Continuous Actuator
  - Set of Discrete Actuators
- SNAC
  - Using Finite Difference
  - Using Proper Orthogonal Decomposition (POD)
- Examples
Example

Consider the one-dimensional diffusion equation
\[ \frac{\partial x(t, y)}{\partial t} = \frac{\partial^2 x(t, y)}{\partial y^2} + u(t, y) \]
with initial condition \( x(t_0 = 0, y) = x_0(y) \),
and \( \frac{\partial x(t, y)}{\partial y} = 0 \) at \( y = 0 \) and \( y = y_f \).

Task: To find control \( u(t, y) \), which minimizes the cost function
\[ J = \frac{1}{2} \int_0^{y_f} \int_0^{t_f} \left[ q x^2(t, y) + r u^2(t, y) \right] dt \, dy \]
Example: Approximation of System Dynamics

Central difference formula:
\[
\frac{\partial^2(x(t,y))}{\partial y^2} \approx \frac{x_{i+1}(t) - 2x_i(t) + x_{i-1}(t)}{(\Delta y)^2}
\]
where \(x_{i+1}(t) = x(t, y + \Delta y), x_i(t) = x(t, y), x_{i-1}(t) = x(t, y - \Delta y)\)

Notation: \(\frac{\partial x(t,y)}{\partial t} = \dot{x}_i(t)\) where \(i = 1, 2, \ldots, n\)

Then \(\dot{x}_i(t) = \frac{x_{i+1}(t) - 2x_i(t) + x_{i-1}(t)}{(\Delta y)^2} + u_i(t)\)
where \(i = 1, 2, \ldots, n\)
Example:
Approximation of System Dynamics

Boundary conditions [to obtain $x_0(t)$ and $x_{n+1}(t)$]:

$$\frac{x_1(t) - x_0(t)}{(\Delta y)} = 0 \quad \text{for} \quad y = 0 \text{ (backward difference at } y = 1)$$

$$\frac{x_{n+1}(t) - x_n(t)}{(\Delta y)} = 0 \quad \text{for} \quad y = n \text{ (forward difference } y = n)$$

This gives

$$x_0(t) = x_1(t)$$
$$x_{n+1}(t) = x_n(t)$$

Example:
Approximation of System Dynamics

$$\dot{x}_1(t) = \frac{1}{(\Delta y)} \left[ x_2(t) - x_1(t) \right] + u_1(t)$$
$$\dot{x}_2(t) = \frac{1}{(\Delta y)} \left[ x_3(t) - 2x_2(t) + x_1(t) \right] + u_2(t)$$
$$\dot{x}_3(t) = \frac{1}{(\Delta y)} \left[ x_4(t) - 2x_3(t) + x_2(t) \right] + u_3(t)$$
$$\vdots$$
$$\dot{x}_{n-1}(t) = \frac{1}{(\Delta y)} \left[ x_n(t) - 2x_{n-1}(t) + x_{n-2}(t) \right] + u_{n-1}(t)$$
$$\dot{x}_n(t) = \frac{1}{(\Delta y)} \left[ -x_n(t) + x_{n-1}(t) \right] + u_n(t)$$
Example:
Approximation of System Dynamics

This can be represented as:
\[ \dot{X}(t) = AX(t) + BU(t), \quad X(0) = X_0 \]
where,
\[
A = \frac{1}{(\Delta y)^2} \begin{bmatrix}
-1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
1 & -2 & 1 & 0 & \cdots & 0 & 0 & 0 \\
0 & 1 & -2 & 1 & \cdots & 0 & 0 & 0 \\
0 & 0 & 1 & -2 & 1 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1
\end{bmatrix}, \quad B = I_{non}
\]

Example:
Approximation of Cost Function

This can be rewritten as
\[ J(t) = \frac{1}{2} \sum_{i=1}^{n-1} \int_0^{t_f} \left[ q x_i^2(t) + r u_i^2(t) \right] dt \]

where \( n \) is the last discretized spatial stage.

This can be rewritten as
\[ J = \frac{1}{2} \int_0^{t_f} \left[ X^T(t)QX(t) + U^T(t)RU(t) \right] dt \]

This is a LQR problem!
Example

Let us consider the following two cases:

Case A: \( t_f = 1.0, \quad y_f = 4.0, \quad B = I, \quad q = r = 1 \)
\[ \Delta t = 0.01, \quad \Delta y = 1.0, \quad \alpha = 1 \]
\[
A = \begin{bmatrix}
-1 & 1 & 0 & 0 & 0 \\
1 & -2 & 1 & 0 & 0 \\
0 & 1 & -2 & 1 & 0 \\
0 & 0 & 1 & -2 & 1 \\
0 & 0 & 0 & 1 & -1 \\
\end{bmatrix}, \quad \alpha = 1
\]
\[ Q = R = \begin{bmatrix}
\frac{1}{2} & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & \frac{1}{2} \\
\end{bmatrix} \]

Case B: \( t_f = 1.0, \quad \Delta t = 0.01, \quad B = I, \quad q = r = 1 \)
\[ y_f = 4.0, \quad \Delta y = 0.5, \quad \alpha = 1 \]
\[
A = 4 \begin{bmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\
\end{bmatrix}, \quad Q = R = \begin{bmatrix}
\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\
\end{bmatrix} \]
Example

Optimal control history at grid points

Comment:
Same result for 5 and 10 grid points

Reference:

Example

Optimal control history at grid points

Comment:
1 grid point solution is good enough

Reference:
Control of a Class of Distributed Parameter Systems Using Optimal Dynamic Inversion

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References


Control of a Class of Distributed Parameter Systems Using Optimal Dynamic Inversion with Continuous Actuator

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Motivation

- Distributed Parameter Systems (DPS) are usually Difficult to control

- Existing techniques:
  - Design-Then-Approximate (e.g. Functional analysis approach)
  - Approximate-Then-Design (e.g. Spatial discretization, Galerkin Projection approach, possibly with POD basis functions)

- New Technique:
  - Falls in D-T-A category without math complexity
  - Dynamic inversion + Variational optimization theory
  - Steady-state convergence can be proved
Problem Description

- System Dynamics:
  \[ \dot{x} = f(x, x', x'', \ldots) + g(x, x', x'', \ldots)u \]
  (with appropriate boundary conditions)

- Goal:
  \[ x(t, y) \to x^*(t, y), \text{ as } t \to \infty \quad \forall \ y \in [0, L] \]

Control Design

- Define error output: 
  \[ z(t) = \frac{1}{2} \int_{0}^{L} [x(t, y) - x^*(t, y)]^2 \text{dy} \]

- Design a controller such that
  \[ \dot{z} + k \ z = 0 \]

- Algebra:
  \[ \int_{0}^{L} (x - x^*) g(x, x', x'', \ldots) u \text{dy} = \gamma \]
  \[ \gamma = -\int_{0}^{L} (x - x^*) [f(x, x', x'', \ldots) - \dot{x}^*] \text{dy} - \frac{k}{2} \int_{0}^{L} (x - x^*)^2 \text{dy} \]
Control Design

- No unique solution: Many solutions exist (scope for optimization)

- Cost Function: 
  \( J = \frac{1}{2} \int_0^L r(y)[u(t, y)]^2 \, dy \)
  (to minimize)

- Augmented Cost Function:
  \[
  \delta \mathcal{J} = \frac{1}{2} \int_0^L r \, dy + \lambda \int_0^L (x - x^*) \, g \, u \, dy - \gamma 
  \]

Control Design

- Necessary condition for optimality:
  \[
  \delta \mathcal{J} = 0 \\
  \int_0^L [ru] \, du + \lambda \int_0^L (x - x^*) \, g \, du \, dy + \delta \lambda \int_0^L (x - x^*) \, g \, u \, dy - \gamma = 0 \\
  ru + \lambda (x - x^*) \, g = 0 \\
  \int_0^L (x - x^*) \, g \, u \, dy = \gamma 
  \]
Control Design: Convergence

- Steady-state control

\[ \dot{x}^* = f^* + g^* u^* \]
\[ u^* = (\dot{x}^* - f^*) \div g^* \]

- Claim:

when \( x(t,y) \rightarrow x^*(t,y) \), \( u(t,y) \rightarrow u^*(t,y) \)

i.e. There is no singularity in the control expression

First we notice that at any point \( y \in (0,L) \), the control solution in Eq.(15) leads to:

\[ u(y) = \frac{-\left[ y(x,y) - x^*(x,y) \right] g(x,y) \left\{ \int f(x,y) - x^*(x,y) \right\} \Phi + \frac{2}{3} \left\{ \left[ x(x,y) - x^*(x,y) \right] \Phi \right\} dy}{r(x,y)} \]

We want to analyze this solution for the case when \( x(t,y) \rightarrow x^*(t,y) \) for all \( y \in [0,L] \). Without loss of generality, we analyze the case in the limit when \( x(t,y) \rightarrow x^*(t,y) \), for \( y \in [y, \sigma/2, \ y + \sigma/2] \subseteq [0,L] \), \( \sigma \rightarrow 0 \) and \( x(t,y) = x^*(t,y) \) everywhere else. In such a limiting case, let us denote \( u(t,y) \) as \( \tilde{u}(t,y) \), which is given by:

\[ \tilde{u}(t,y) = \frac{-\left[ x(t,y) - x^*(t,y) \right] g(t,y) \left\{ \int f(t,y) - x^*(t,y) \right\} \Phi + \frac{2}{3} \left\{ \left[ x(t,y) - x^*(t,y) \right] \Phi \right\} dy}{r(t,y)} \]

\[ = \frac{-\left[ x(t,y) - x^*(t,y) \right] g(t,y) \left\{ \int f(t,y) - x^*(t,y) \right\} \Phi + \frac{2}{3} \left\{ \left[ x(t,y) - x^*(t,y) \right] \Phi \right\} dy}{r(t,y)} \]

Moreover, this happens \( \forall y \in (0,L) \). Hence \( u(t,y) \rightarrow u^*(t,y) \) as \( x(t,y) \rightarrow x^*(t,y) \), \( \forall y \in [0,L] \).
Control Design

- Solve for the control variable:

\[
 u = \frac{\gamma (x - x^*) g}{r(y) \int_0^L (x - x^*)^2 g^2 r(y) \ dy} \quad \text{Note: Control Solution is in “Closed Form”}
\]

- Special Case:

\[
 r(y) = c \in \mathbb{R}^+, \quad g(x, x', x'', ...) = \beta \in \mathbb{R}
\]

\[
 u = \frac{\gamma (x - x^*)}{\beta \int_0^L (x - x^*)^2 \ dy}
\]

Control Design: Final Expression

\[
 u^* = \begin{cases} 
 -\frac{1}{g} \left[f^* - \dot{x}^*\right], & \text{if } x(t, y) = x^*(t, y) \ \forall y \in [0, L] \\
 \frac{\gamma (x - x^*) g}{r(y) \int_0^L (x - x^*)^2 g^2 r(y) \ dy}, & \text{otherwise}
\end{cases}
\]
A Motivating Problem

- Heat transfer in a fin
- Temperature dynamics

\[ \frac{\partial T}{\partial t} = \alpha_l \left( \frac{\partial^2 T}{\partial y^2} \right) + \alpha_i (T - T_a) + \alpha_e (T_e - T_m) + \beta S \]

- Boundary conditions
- Desired temperature profile
  \[ T(y) = T_0 + (T_m - T_0) \frac{y}{L} \]
- Control gain
  \[ k = 1/\tau, \quad \tau = 30 \text{sec} \]

Numerical Results:
Sinusoidal initial condition

Temperature history  Control history
Numerical Results:
Sinusoidal initial condition

Deviated temperature profile history

Comparison of Steady-state control

Numerical Results:
Random initial condition

Temperature history

Control history
Numerical Results:
Random initial condition

Deviated temperature profile history

Comparison of Steady-state control

Control of a Class of Distributed Parameter Systems Using Optimal Dynamic Inversion with a Set of Discrete Actuators

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### Problem Description

- **System Dynamics:**
  \[ \dot{x} = f(x, x', x'', \ldots) + \sum_{a=1}^{M} g(x, x', x'', \ldots) \tilde{u}_a \]

- **Control Structure:**
  - Inside interval \( \tilde{u}_a = const. \)
  - Outside interval \( \tilde{u}_a = 0 \)
  - No overlapping of different controllers
  - No boundary control

- **Goal:**
  \[ x(t, y) \to x^*(t, y), \quad \text{as} \quad t \to \infty \quad \text{FOR ALL} \quad y \in [0, L] \]

### Control Design

- **Define Output:**
  \[ z(t) = \frac{1}{2} \int_0^t [x(t, y) - x^*(t, y)]^2 \, dy \]

- **Design a controller such that**
  \[ \ddot{z} + k \, z = 0 \]

- **Algebra:**
  \[ \int_0^t (x - x^*) \, g(x, x', x'', \ldots) \, u \, dy = \gamma \]
  \[ \gamma \triangleq -\int_0^t (x - x^*) \left[ f(x, x', x'', \ldots) - \dot{x}^* \right] dy - \frac{k}{2} \int_0^t (x - x^*)^2 \, dy \]
Control Design....Contd.

- **Constraint Eq.**:
  \[ I_1 \bar{u}_1 + \cdots + I_M \bar{u}_M = \gamma \]
  \[ I_m = \int_{x_m - \frac{\eta}{2}}^{x_m + \frac{\eta}{2}} (x - x_m^*) \, g \, dy, \quad m = 1, \ldots, M \]

- **Cost Function**:
  \[ J = \frac{1}{2} \left( r_1 w_1 \bar{u}_1^2 + \cdots + r_m w_m \bar{u}_m^2 \right) \]
  (minimize)

- **Final Solution**:
  \[ \bar{u}_m = \frac{I_m \gamma}{r_m w_m \sum_{m=1}^{M} I_m^2 / (r_m w_m)}, \quad m = 1, \ldots, M \]
  (closed-form expression)

- **Special Case**:
  \[ \pi_n = \frac{I_n \gamma}{\|I_n\|}, \quad I \triangleq \begin{bmatrix} I_1 & \cdots & I_m \end{bmatrix}, \quad r_n w_n = \cdots = r_m w_m \]

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Control Design....Contd.

- **Singularity Problem**:
  \[ \bar{u}_m \to \infty \quad \text{as all} \quad I_1, \ldots, I_M \to 0 \quad \text{and} \quad \gamma \gg 0 \]

- **Revised Goal**:
  \[ X \triangleq \begin{bmatrix} x_1, \cdots, x_m \end{bmatrix} \to X^* \triangleq \begin{bmatrix} x_1^*, \cdots, x_m^* \end{bmatrix} \]
  (objective is achieved at the node points only where the controllers are located)
  \[ E \triangleq (X - X^*) \to 0 \quad \text{as} \quad t \to \infty \]

- **Design a control s.t.**
  \[ \dot{E} + KE = 0, \quad K > 0 \]
Control Design....Contd.

- Gain $K$ chosen as diagonal with: $k_m = (1/\tau_m)$

- $m^{th}$ channel equation: $\dot{e}_m + k_m e_m = 0$

- Finally: $\bar{u}_m = \frac{1}{g_m} \left[ \dot{x}_m^* - f_m - k_m (x_m - x_m^*) \right]$

$$x_m \triangleq x(t, y_m) \quad f_m \triangleq f(t, y_m)$$

$$x_m^* \triangleq x^*(t, y_m) \quad g_m \triangleq g(t, y_m)$$

Final control solution (for implementation)

$$\bar{u}_m = \begin{cases} \frac{1}{g_m} \left[ \dot{x}_m^* - f_m - k_m (x_m - x_m^*) \right], & \text{if } \| I \|_2 < \text{tol} \\ \frac{I_m \gamma}{r_m W_m \sum_{\tilde{m}} \frac{T_{\tilde{m}}^2}{(r_{\tilde{m}} W_{\tilde{m}})}}, & \text{otherwise} \end{cases}$$
A Motivating Problem

Mathematical model

- Heat transfer in a fin

- Temperature dynamics

\[ \frac{\partial T}{\partial t} = \alpha_1 \left( \frac{\partial^2 T}{\partial y^2} \right) + \alpha_2 (T - T_m) + \alpha_3 (T^4 - T_m^4) + \beta \sum_{n=1}^{M} \delta_n \]

- Boundary conditions

\[ T_{y=0} = T_m, \quad \frac{\partial T}{\partial y} \bigg|_{y=L} = 0 \]

- Desired temperature profile

\[ T^*(y) = T_m + (T_m - T_0) e^{-y} \]

Numerical Results:

Sinusoidal initial condition

Temperature history | Control history

Wavy steady state is because of insufficient number of controllers
Numerical Results:
Sinusoidal initial condition

Temperature history
Control history

Wavy steady state is no more there!

Numerical Results:
Random initial condition

Temperature history
Control history

Similar results have been obtained from numerous random initial conditions!
Conclusions: ODI Approach

- A new technique for nonlinear DPS control:
  - Dynamic inversion (DI) + Optimization theory
  - Largely driven by DI; has some optimization feature too
- Falls in the “design-then-approximate” philosophy without the math complexity
- **Closed form solution: No computational complexity**
- Continuous actuator case: Theoretical elegance
  (Guaranteed convergence to steady-state expression and no singularity problem)
- Discrete actuators case: Practical relevance
- Successfully demonstrated in a challenging problem

Thanks for the Attention....!!