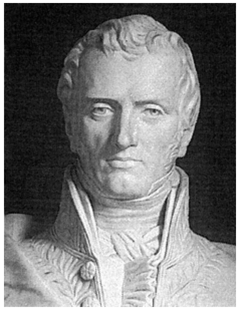
Module 6

Lecture 2: Navier-Stokes and Saint Venant equations

Navier-Stokes Equations



Claude-Louis Navier

Sir George Gabriel Stokes

- St.Venant equations are derived from Navier-Stokes Equations for shallow water flow conditions.
- The Navier-Stokes Equations are a general model which can be used to model water flows in many applications.
- A general flood wave for 1-D situation can be described by the Saint-Venant equations.

Navier-Stokes Equations

- It consists of 4 nonlinear PDE of mixed hyperbolic-parabolic type describing the fluid hydrodynamics in 3D.
- Expression of **F**=m**a** for a fluid in a differential volume

$$a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \qquad (6.6)$$

$$a_y = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \qquad (6.7)$$

$$a_z = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} \qquad (6.8)$$

$$a_i = \frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j}$$

where i: x, y, z
u_i: u, v, w
u_j: u, v, w

- The acceleration vector contains local and convective acceleration terms

Navier-Stokes Equations **Contd…**

- The force vector is broken into a surface force and a body force per unit volume.
- The body force vector is due only to gravity while the pressure forces and the viscous shear stresses make up the surface forces(i.e. per unit mass).

$$f_x = g_x + \frac{1}{\rho}\left[-\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \right] \quad (6.9)$$

$$f_y = g_y + \frac{1}{\rho}\left[-\frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} \right] \quad (6.10)$$

$$f_z = g_z + \frac{1}{\rho}\left[-\frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} \right] \quad (6.11)$$

- The stresses are related to fluid element displacements by invoking the Stokes viscosity law for an incompressible fluid.

$$\tau_{xx} = 2\mu\frac{\partial u}{\partial x},\ \tau_{yy} = 2\mu\frac{\partial v}{\partial x},\ \tau_{zz} = 2\mu\frac{\partial w}{\partial x} \qquad (6.12)$$

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \qquad (6.13)$$

$$\tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \qquad (6.14)$$

$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \qquad (6.15)$$

Navier-Stokes Equations

- Substituting eqs. 6.12-6.15 into eqs. 6.9-6.11, we get,

$$f_x = g_x - \frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right] \quad (6.16)$$

$$f_y = g_y - \frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right] \quad (6.17)$$

$$f_z = g_z - \frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right] \quad (6.18)$$

$$f_i = g_i - \frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu\frac{\partial^2 u_i}{\partial x_j \partial x_j} \qquad Einstein \;\; notation$$

Navier-Stokes Equations

Contd…

- The three N-S momentum equations can be written in compact form as

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = \frac{-1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j} + g_i \qquad (6.19)$$

- The equation of continuity for an incompressible fluid

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\frac{\partial u_i}{\partial x_i} = 0 \qquad (6.20)$$

Saint Venant Equations

- **The Saint Venant Equations** were formulated in the 19th century by two mathematicians, **de Saint Venant** and **Bousinnesque**.

- The solution of the St. Venant equations is known as dynamic routing, which is generally the standard to which other methods are measured or compared.

Joseph Valentin Boussinesq

Jean Claude Saint-Venant

Continuity equation:

$$\frac{\partial Q}{\partial x} + \frac{\partial A}{\partial t} = 0$$

Momentum equation:

$$\frac{1}{A}\frac{\partial Q}{\partial t} + \frac{1}{A}\frac{\partial}{\partial x}\left(\frac{Q^2}{A}\right) + g\frac{\partial y}{\partial x} - g(S_o - S_f) = 0$$

Q-Discharge through the channel
A-Area of cross-section of flow
y- Depth of flow
S_0-Channel bottom slope
S_f- Friction slope

Assumptions of St. Venant Equations

- Flow is one-dimensional
- Hydrostatic pressure prevails and vertical accelerations are negligible
- Streamline curvature is small.
- Bottom slope of the channel is small.
- Manning's and Chezy's equation are used to describe resistance effects
- The fluid is incompressible
- Channel boundaries are considered fixed and therefore not susceptible to erosion or deposition.

1D gradually varied unsteady flow in an open channel is given by St. Venant equations:

- Continuity Equation (based on Conservation of Mass)
- Momentum Equation (based on Conservation of Momentum)

1-D Open channel flow

In the diagrams given,

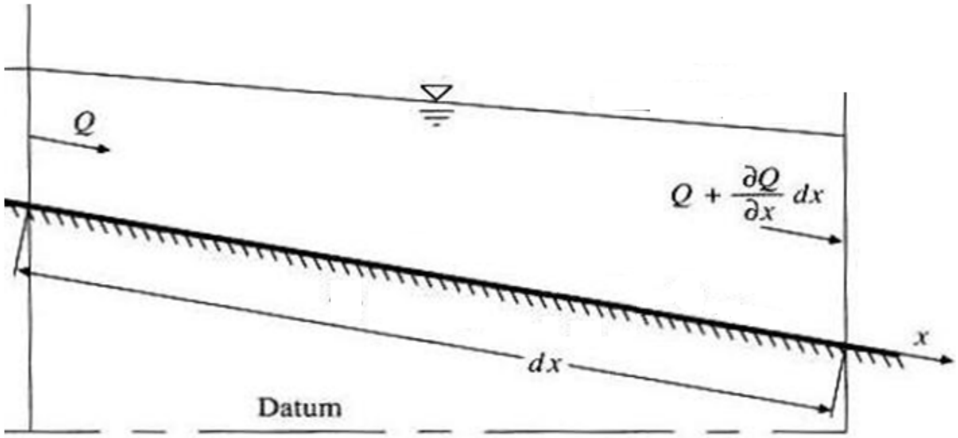
Q = inflow to the control volume

q = lateral inflow

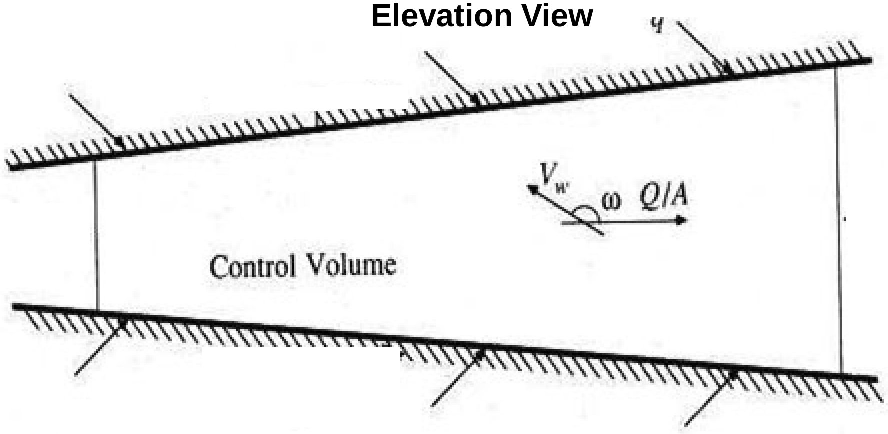
$\frac{\partial Q}{\partial x}$ = Rate of change of flow with distance

$Q + \frac{\partial Q}{\partial x} dx$ = Outflow from the C.V.

$\frac{\partial(\rho A dx)}{\partial t}$ = Change in mass



Elevation View



Plan View

St. Venant equations

Continuity equation:

$$\frac{\partial Q}{\partial x} + \frac{\partial A}{\partial t} = 0$$

Q-Discharge through the channel
A-Area of cross-section of flow

Conservation of Mass

- In any control volume consisting of the fluid (water) under consideration, the net change of mass in the control volume due to inflow and outflow is equal to the net rate of change of mass in the control volume

Continuity Equation-Derivation

Q = AV = volume water discharge [L^3/T]
ρQ = Mass water discharge = ρAV [M/T]

∂/∂t(Mass in control volume) = Net mass inflow rate (assuming q=0)

$$\frac{\partial(\rho A)}{\partial t}\Delta x = \rho AV\Big|_{x} - \rho AV\Big|_{x+\Delta x} = -\rho\frac{\partial(AV)}{\partial x}\Delta x$$

$$i.e\ \ \frac{\partial(\rho A)}{\partial t}\Delta x + \rho\frac{\partial(AV)}{\partial x}\Delta x = 0$$

$$\Rightarrow \rho\Delta x\left(\frac{\partial A}{\partial t} + \frac{\partial AV}{\partial x}\right) = 0;\ \ Here\ AV = Q,\ \ discharge$$

$$through\ the\ cross-section$$

$$\Rightarrow \left(\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x}\right) = 0$$

In 1-D open channel flow continuity equation becomes,

Conservation form

$$\frac{\partial Q}{\partial x} + \frac{\partial A}{\partial t} - q = 0$$

Non-conservation form

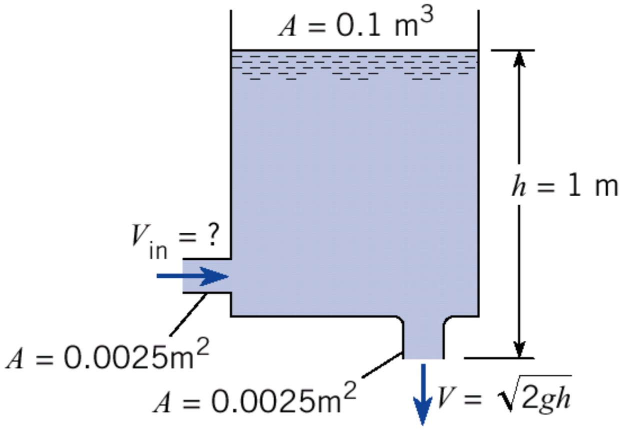
(velocity is dependent variable)

$$\frac{\partial (Vy)}{\partial x} + \frac{\partial y}{\partial t} = 0$$

$$V\frac{\partial y}{\partial x} + y\frac{\partial V}{\partial x} + \frac{\partial y}{\partial t} = 0$$

Example Problem

Calculate the inlet velocity Vin from the diagram shown.



$$0 = \frac{d}{dt}\int_{CV} \rho d\forall + \sum_{CS} \rho \vec{\mathbf{V}} \cdot \vec{\mathbf{A}}$$

$$= \frac{d}{dt}(\rho A_{tank} h) - \rho V_{in} A_{in} + \rho V_{out} A_{out}$$

$$= A_{tank}\frac{dh}{dt} - V_{in} A_{in} + V_{out} A_{out}$$

$$= 0.1 * 0.1x10^{-2} - V_{in}(0.0025) + \sqrt{2g * 1}(0.0025)$$

$$V_{in} = 4.47\, m/s$$

Momentum

In mechanics, as per **Newton's 2nd Law:**

Net force = time rate of change of momentum

Change in momentum in the s direction

$$\sum F_s = \Delta(mv_s)$$

Sum of forces in the s direction

mass

Velocity in the s direction

Momentum Equation

➔ The change in momentum of a body of water in a flowing channel is equal to the resultant of all the external forces acting on that body.

$$\sum F = \underbrace{\frac{d}{dt}\iiint_{c.v.} V\rho d\forall}_{\text{Momentum stored within the C.V}} + \underbrace{\iint_{c.s.} V\rho V.dA}_{\text{Momentum flow across the C. S.}}$$

Sum of forces on the C.V.

Momentum stored within the C.V

Momentum flow across the C. S.

Conservation of Momentum

- This law states that the rate of change of momentum in the control volume is equal to the net forces acting on the control volume
- Since the water under consideration is moving, it is acted upon by external forces which will lead to the Newton's second law

$$\sum F = \frac{d}{dt}\iiint_{c.v.} V\rho\, d\forall + \iint_{c.s.} V\rho V.dA$$

Sum of forces on the C.V.

Momentum stored within the C.V

Momentum flow across the C. S.

$$\frac{1}{A}\frac{\partial Q}{\partial t} + \frac{1}{A}\frac{\partial}{\partial x}\left(\frac{Q^2}{A}\right) + g\frac{\partial y}{\partial x} - g(S_o - S_f) = 0$$

Applications of different forms of momentum equation

- Kinematic wave: when gravity forces and friction forces balance each other (steep slope channels with no back water effects)

- Diffusion wave: when pressure forces are important in addition to gravity and frictional forces

- Dynamic wave: when both inertial and pressure forces are important and backwater effects are not negligible (mild slope channels with downstream control)

Approximations to the full dynamic equations

The three most common approximations or simplifications are:

- Kinematic
- Diffusion
- Quasi-steady models

Kinematic wave routing:

- Assumes that the motion of the hydrograph along the channel is controlled by gravity and friction forces. Therefore, uniform flow is assumed to take place. Then momentum equation becomes a wave equation:

$$\frac{\partial Q}{\partial t} + c\frac{\partial Q}{\partial x} = 0$$

where Q is the discharge, t the time, x the distance along the channel, and c the celerity of the wave (speed).

➔ A kinematic wave travels downstream with speed c without experiencing any attenuation or change in shape. Therefore, diffusion is absent.

Diffusion wave routing

- The diffusion wave approximation includes the pressure differential term but still considers the inertial terms negligible; this constitutes an improvement over the kinematic wave approximation.

$$S_f = S_0 - \frac{\partial y}{\partial x}$$

- The pressure differential term allows for diffusion (attenuation) of the flood wave and the inclusion of a downstream boundary condition which can account for backwater effects.

- This is appropriate for most natural, slow-rising flood waves but may lead to problems for flash flood or dam break waves

Quasi-Steady Dynamic Wave Routing

- It incorporates the convective acceleration term but not the local acceleration term, as indicated below:

$$S_f = S_0 - \left(\frac{\partial y}{\partial x}\right) - \left(\frac{V \partial V}{g \partial x}\right)$$

- In channel routing calculations, the convective acceleration term and local acceleration term are opposite in sign and thus tend to negate each other. If only one term is used, an error results which is greater in magnitude than the error created if both terms were excluded (Brunner, 1992).
- Therefore, the quasi-steady approximation is not used in channel routing.