Analysis of Variance and Design of Experiments-I

MODULE - I

LECTURE - 2

SOME RESULTS ON LINEAR ALGEBRA, MATRIX THEORY AND DISTRIBUTIONS

Dr. Shalabh
Department of Mathematics and Statistics
Indian Institute of Technology Kanpur
Quadratic forms

If $A$ is a given matrix of order $m \times n$ and $X$ and $Y$ are two given vectors of order $m \times 1$ and $n \times 1$ respectively, then the quadratic form is given by

$$X'AY = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} x_i y_j,$$

where $a_{ij}$'s are the nonstochastic elements of $A$.

If $A$ is square matrix of order $m$ and $X=Y$, then

$$X'AX = a_{11}x_1^2 + \ldots + a_{mm}x_m^2 + (a_{12} + a_{21})x_1x_2 + \ldots + (a_{m-1,m} + a_{m,m-1})x_{m-1}x_m.$$

If $A$ is symmetric also, then

$$X'AX = a_{11}x_1^2 + \ldots + a_{mm}x_m^2 + 2a_{12}x_1x_2 + \ldots + 2a_{m-1,m}x_{m-1}x_m$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}x_i x_j,$$

is called a quadratic form in $m$ variables $x_1, x_2, \ldots, x_m$ or a quadratic form in $X$.

- To every quadratic form corresponds a symmetric matrix and vice versa.
- The matrix $A$ is called the matrix of quadratic form.
- The quadratic form $X'AX$ and the matrix $A$ of the form is called
  - Positive definite if $X'AX > 0$ for all $x \neq 0$.
  - Positive semi definite if $X'AX \geq 0$ for all $x \neq 0$.
  - Negative definite if $X'AX < 0$ for all $x \neq 0$.
  - Negative semi definite if $X'AX \leq 0$ for all $x \neq 0$. 
• If $A$ is positive semi definite matrix then $a_{ii} \geq 0$ and if $a_{ii} = 0$ then $a_{ij} = 0$ for all $j$, and $a_{ji} = 0$ for all $j$.

• If $P$ is any nonsingular matrix and $A$ is any positive definite matrix (or positive semi-definite matrix) then $P^T AP$ is also a positive definite matrix (or positive semi-definite matrix).

• A matrix $A$ is positive definite if and only if there exists a non-singular matrix $P$ such that $A = P^T P$.

• A positive definite matrix is a nonsingular matrix.

• If $A$ is $m \times n$ matrix and $\text{rank}(A) = m < n$ then $AA^T$ is positive definite and $A^TA$ is positive semidefinite.

• If $A$ $m \times n$ matrix and $\text{rank}(A) = k < m < n$, then both $A^TA$ and $AA^T$ are positive semidefinite.
Simultaneous linear equations

The set of \( m \) linear equations in \( n \) unknowns \( x_1, x_2, \ldots, x_n \) and scalars \( a_{ij} \) and \( b_j \), \( i = 1, 2, \ldots, m, j = 1, 2, \ldots, n \) of the form

\[
\begin{align*}
  a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\
  a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\
  &\vdots \\
  a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m
\end{align*}
\]

can be formulated as

\[
AX = b
\]

where \( A \) is a real matrix of known scalars of order \( m \times n \) called as coefficient matrix, \( X \) is real vector and \( b \) is \( n \times 1 \) real vector of known scalars given by

\[
A = \begin{pmatrix}
  a_{11} & a_{12} & \cdots & a_{1n} \\
  a_{21} & a_{22} & \cdots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix},
\]

is an \( m \times n \) real matrix called as coefficient matrix,

\[
X = \begin{pmatrix}
  x_1 \\
  x_2 \\
  \vdots \\
  x_n
\end{pmatrix},
\]

is an \( n \times 1 \) vector of variables and

\[
b = \begin{pmatrix}
  b_1 \\
  b_2 \\
  \vdots \\
  b_m
\end{pmatrix},
\]

is an \( m \times 1 \) real vector.
• If $A$ is an $n \times n$ nonsingular matrix, then $AX = b$ has a unique solution.

• Let $B = [A, b]$ is an augmented matrix. A solution to $AX = b$ exist if and only if $\text{rank}(A) = \text{rank}(B)$.

• If $A$ is an $m \times n$ matrix of rank $m$, then $AX = b$ has a solution.

• Linear homogeneous system $AX = 0$ has a solution other than $X = 0$ if and only if $\text{rank}(A) < n$.

• If $AX = b$ is consistent then $AX = b$ has a unique solution if and only if $\text{rank}(A) = n$.

• If $a_{ii}$ is the $i^{th}$ diagonal element of an orthogonal matrix, then $-1 \leq a_{ii} \leq 1$.

• Let the $n \times n$ matrix be partitioned as $A = [a_1, a_2, \ldots, a_n]$ where $a_i$ is an $n \times 1$ vector of the elements of $i^{th}$ column of $A$.

  A necessary and sufficient condition that $A$ is an orthogonal matrix is given by the following:

  (i) $a_i' a_i = 1$ for $i = 1, 2, \ldots, n$

  (ii) $a_i' a_j = 0$ for $i \neq j = 1, 2, \ldots, n$.

---

### Orthogonal matrix

A square matrix $A$ is called an orthogonal matrix if $A' A = AA' = I$ or equivalently if $A^{-1} = A'$.

• An orthogonal matrix is non-singular.

• If $A$ is orthogonal, then $AA'$ is also orthogonal.

• If $A$ is an $n \times n$ matrix and let $P$ is an $n \times n$ orthogonal matrix, then the determinants of $A$ and $P'AP$ are the same.
Random vectors

Let $Y_1, Y_2, ..., Y_n$ be $n$ random variables then $Y = (Y_1, Y_2, ..., Y_n)'$ is called a random vector.

- The mean vector of $Y$ is
  $$E(Y) = ((E(Y_1), E(Y_2), ..., E(Y_n))'.$$

- The covariance matrix or dispersion matrix of $Y$ is
  $$Var(Y) = \begin{pmatrix}
  Var(Y_1) & Cov(Y_1, Y_2) & ... & Cov(Y_1, Y_n) \\
  Cov(Y_2, Y_1) & Var(Y_2) & ... & Cov(Y_2, Y_n) \\
  \vdots & \vdots & \ddots & \vdots \\
  Cov(Y_n, Y_1) & Cov(Y_n, Y_2) & ... & Var(Y_n)
  \end{pmatrix}$$
  which is a symmetric matrix.

- If $Y_1, Y_2, ..., Y_n$ are pair-wise uncorrelated, then the covariance matrix is a diagonal matrix.

- If $Var(Y_i) = \sigma^2$ for all $i = 1, 2, ..., n$ then $Var(Y) = \sigma^2 I_n$. 
### Linear function of random variable

If \( Y_1, Y_2, ..., Y_n \) are \( n \) random variables and \( k_1, k_2, ..., k_n \) are scalars, then \( \sum_{i=1}^{n} k_iY_i \) is called a linear function of random variables \( Y_1, Y_2, ..., Y_n \).

If \( Y = (Y_1, Y_2, ..., Y_n)' \), \( K = (k_1, k_2, ..., k_n)' \) then \( K'Y = \sum_{i=1}^{n} k_iY_i \),

- the mean \( K'Y \) is \( E(K'Y) = K'E(Y) = \sum_{i=1}^{n} k_iE(Y_i) \) and
- the variance of \( K'Y \) is \( Var(K'Y) = K'Var(Y)K \).

### Multivariate normal distribution

A random vector \( Y = (Y_1, Y_2, ..., Y_n)' \) has a multivariate normal distribution with mean vector \( \mu = (\mu_1, \mu_2, ..., \mu_n) \) and dispersion matrix \( \Sigma \) if its probability density function is

\[
 f(Y | \mu, \Sigma) = \frac{1}{(2\pi)^{n/2} |\Sigma|^{1/2}} \exp \left[ -\frac{1}{2} (Y - \mu)' \Sigma^{-1} (Y - \mu) \right] 
\]

assuming \( \Sigma \) is a nonsingular matrix.
Chi-square distribution

- If $Y_1, Y_2, ..., Y_k$ are identically and independently distributed random variables following the normal distribution with common mean 0 and common variance 1, then the distribution of $\sum_{i=1}^{k} Y_i^2$ is called the $\chi^2$-distribution with $k$ degrees of freedom.

- The probability density function of $\chi^2$-distribution with $k$ degrees of freedom is given as

$$f_{\chi^2}(x) = \frac{1}{\Gamma(k/2)2^{k/2}} x^{k-1} \exp \left( -\frac{x}{2} \right); \quad 0 < x < \infty.$$ 

- If $Y_1, Y_2, ..., Y_k$ are independently distributed following the normal distribution with common means 0 and common variance $\sigma^2$, then $\frac{1}{\sigma^2} \sum_{i=1}^{k} Y_i^2$ has $\chi^2$-distribution with $k$ degrees of freedom.

- If the random variables $Y_1, Y_2, ..., Y_k$ are normally distributed with non-null means $\mu_1, \mu_2, ..., \mu_k$ but common variance 1, then the distribution of $\sum_{i=1}^{k} Y_i^2$ has non-central $\chi^2$-distribution with $k$ degrees of freedom and non-centrality parameter $\lambda = \sum_{i=1}^{k} \mu_i^2$.

- If $Y_1, Y_2, ..., Y_k$ are independently distributed following the normal distribution with means $\mu_1, \mu_2, ..., \mu_k$ but common variance $\sigma^2$ then $\frac{1}{\sigma^2} \sum_{i=1}^{k} Y_i^2$ has non-central $\chi^2$-distribution with $k$ degrees of freedom and noncentrality parameter $\lambda = \frac{1}{\sigma^2} \sum_{i=1}^{k} \mu_i^2$. 
• If $U$ has a Chi-square distribution with $k$ degrees of freedom then $E(U) = k$ and $\text{Var}(U) = 2k$.

• If $U$ has a noncentral Chi-square distribution with $k$ degrees of freedom and noncentrality parameter $\lambda$ then $E(U) = k + \lambda$ and $\text{Var}(U) = 2k + 4\lambda$.

• If $U_1, U_2, \ldots, U_k$ are independently distributed random variables with each $U_i$ having a noncentral Chi-square distribution with $n_i$ degrees of freedom and noncentrality parameter $\lambda_i$, $i = 1, 2, \ldots, k$ then $\sum_{i=1}^{k} U_i$ has noncentral Chi-square distribution with $\sum_{i=1}^{k} n_i$ degrees of freedom and noncentrality parameter $\sum_{i=1}^{k} \lambda_i$.

• Let $X = (X_1, X_2, \ldots, X_n)'$ has a multivariate distribution with mean vector $\mu$ and positive definite covariance matrix $\Sigma$. Then $X'AX$ is distributed as noncentral $\chi^2$ with $k$ degrees of freedom if and only if $\Sigma A$ is an idempotent matrix of rank $k$.

• Let $X = (X_1, X_2, \ldots, X_n)$ has a multivariate normal distribution with mean vector $\mu$ and positive definite covariance matrix $\Sigma$. Let the two quadratic forms-

$\quad X' A_1 X$ is distributed as $\chi^2$ with $n_1$ degrees of freedom and noncentrality parameter $\mu' A_1 \mu$ and

$\quad X' A_2 X$ is distributed as $\chi^2$ with $n_2$ degrees of freedom and noncentrality parameter $\mu' A_2 \mu$.

Then $X' A_1 X$ and $X' A_2 X$ are independently distributed if $A_1 \Sigma A_2 = 0.$
**t- distribution**

If

- $X$ has a normal distribution with mean 0 and variance 1,
- $Y$ has a $\chi^2$ distribution with $n$ degrees of freedom, and
- $X$ and $Y$ are independent random variables,

then the distribution of the statistic $T = \frac{X}{\sqrt{Y/n}}$ is called the $t$-distribution with $n$ degrees of freedom.

The probability density function of $T$ is

$$f_T(t) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\sqrt{n\pi}} \left(1 + \frac{t^2}{n}\right)^{-\frac{n+1}{2}}; \quad -\infty < t < \infty.$$ 

- If the mean of $X$ is non zero then the distribution of $\frac{X}{\sqrt{Y/n}}$ is called the noncentral $t$-distribution with $n$ degrees of freedom and noncentrality parameter $\mu$.  

**F-distribution**

- If $X$ and $Y$ are independent random variables with $\chi^2$-distribution with $m$ and $n$ degrees of freedom respectively, then the distribution of the statistic $F = \frac{X/m}{Y/n}$ is called the F-distribution with $m$ and $n$ degrees of freedom. The probability density function of $F$ is

$$f_F(f) = \frac{\Gamma\left(\frac{m+n}{2}\right)\left(\frac{m}{n}\right)^{m/2}}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)} f^{\left(\frac{m-2}{2}\right)} \left(1 + \left(\frac{m}{n}\right)f\right)^{-\left(\frac{m+n}{2}\right)}; \quad 0 < f < \infty.$$  

- If $X$ has a noncentral Chi-square distribution with $m$ degrees of freedom and noncentrality parameter $\lambda$; $Y$ has a $\chi^2$ distribution with $n$ degrees of freedom, and $X$ and $Y$ are independent random variables, then the distribution of $F = \frac{X/m}{Y/n}$ is the noncentral F distribution with $m$ and $n$ degrees of freedom and noncentrality parameter $\lambda$. 