Analysis of Variance and Design of Experiment-I

MODULE – V

LECTURE - 25

FACTORIAL EXPERIMENTS

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Factorial experiments involve simultaneously more than one factor each at two or more levels. Several factors affect simultaneously the characteristic under study in factorial experiments and the experimenter is interested in the main effects and the interaction effects among different factors.

First we consider an example to understand the utility of factorial experiments.

**Example:** Suppose the yield from different plots in an agricultural experiment depend upon

1. (i) variety of crop and  
   (ii) type of fertilizer.

Both the factors are in the control of experimenter.

2. (iii) Soil fertility. This factor is not in the control of experimenter.

In order to compare the different crop varieties

- assign it to different plots keeping other factors like irrigation, fertilizer, etc. fixed and the same for all the plots.

- The conclusions for this will be valid only for the crops grown under similar conditions with respect to the factors like fertilizer, irrigation etc.
In order to compare different fertilizers (or different dosage of fertilizers)
• sow single crop on all the plots and vary the quantity of fertilizer from plot to plot.
• The conclusions will become invalid if different varieties of crop are sown.
• It is quite possible that one variety may respond differently than another to a particular type of fertilizer.

Suppose we wish to compare,
- two crop varieties – a and b, keeping the fertilizer fixed and
- three varieties of fertilizers – A, B and C.

This can be accomplished with two randomized block designs (RBD) by assigning the treatments at random to three plots in any block and two crop varieties at random.

The possible arrangement of the treatments may appear as follows.

```
bB  bA  bC  
|  bC  bB  bA  |
|  bA  bC  bB  |
```

```
aA  aB  aC  
|  aC  aA  aB  |
|  aB  aC  aA  |
```

With these two RBDs,
- the difference among two fertilizers can be estimated
- but the difference among the crop varieties cannot be estimated. The difference among the crop varieties is entangled with the difference in blocks.
On the other hand, if we use three sets of three blocks each and each block having two plots, then

- randomize the varieties inside each block and
- assign treatments at random to three sets.

The possible arrangement of treatment combinations in blocks can be as follows:

<table>
<thead>
<tr>
<th>bB</th>
<th>aB</th>
<th>aC</th>
<th>bC</th>
</tr>
</thead>
<tbody>
<tr>
<td>aB</td>
<td>bB</td>
<td>bC</td>
<td>aC</td>
</tr>
<tr>
<td>bB</td>
<td>aB</td>
<td>aC</td>
<td>bC</td>
</tr>
</tbody>
</table>

and

<table>
<thead>
<tr>
<th>aA</th>
<th>bA</th>
<th>bA</th>
<th>aA</th>
</tr>
</thead>
<tbody>
<tr>
<td>aA</td>
<td>bA</td>
<td>bA</td>
<td>aA</td>
</tr>
</tbody>
</table>

Here the difference between crop varieties is estimable but the difference between fertilizer treatment is not estimable.

Factorial experiments overcome this difficulty and combine each crop with each fertilizer treatment. There are six treatment combinations as

\[ aA, aB, aC, bA, bB, bC. \]
Keeping the total number of observations to be 18 (as earlier), we can use RBD with three blocks with six plots each, e.g.

```
 bA   aC   aB   bB   aA   bC
 aA   aC   bC   aB   bB   bA
 bB   aB   bA   aC   aA   bC
```

Now we can estimate the
- difference between crop varieties and
- difference between fertilizer treatments.

Factorial experiments involves simultaneously more than one factor each at two or more levels.

If the number of levels for each factor is the same, we call it as **symmetrical factorial experiment**.

If the number of levels of each factor is not the same, then we call it as a **symmetrical or mixed factorial experiment**.

We consider only symmetrical factorial experiments.
Through the factorial experiments, we can study the individual effect of each factor and interaction effect.

Now we consider a $2^2$ factorial experiment with an example and try to develop and understand the theory and notations through this example.

A general notation for representing the factors is to use capital letters, e.g., $A, B, C$ etc. and levels of a factor are represented in small letters.

For example, if there are two levels of $A$, they are denoted as $a_0$ and $a_1$. Similarly the two levels of $B$ are represented as $b_0$ and $b_1$. Other alternative representation to indicate the two levels of $A$ is 0 (for $a_0$) and 1 (for $a_1$).

The factors of $B$ are then 0 (for $b_0$) and 1 (for $b_1$).

**Note:** An important point to remember is that the factorial experiments are conducted in a design of experiment. For example, the factorial experiment is conducted as an RBD.
Factorial experiments with factors at two levels \((2^2\) factorial experiment\)

Suppose in an experiment, the values of current and voltage in an experiment affect the rotation per minutes \((rpm)\) of a fan speed. Suppose there are two levels of current.

- 5 Ampere, call it as level 1 \((C_1)\) and denote it as \(a_0\)
- 10 Ampere, call it as level 2 \((C_2)\) and denote it as \(a_1\).

Similarly, the two levels of voltage are

- 200 volts, call it as level 1 \((V_0)\) and denote it as \(b_0\)
- 220 volts, call it as level 2 \((V_1)\) and denote it as \(b_1\).

The two factors are denoted as \(A\), say for current and \(B\), say for voltage.

In order to make an experiment, there are 4 different combinations of values of current and voltage.

1. Current = 5 Ampere and Voltage = 200 Volts, denoted as \(C_0V_0 \equiv a_0b_0\).
2. Current = 5 Ampere and Voltage = 220 Volts, denoted as \(C_0V_1 \equiv a_0b_1\).
3. Current = 10 Ampere and Voltage = 200 Volts, denoted as \(C_1V_0 \equiv a_1b_0\).
4. Current = 10 Ampere and Voltage = 220 Volts, denoted as \(C_1V_1 \equiv a_1b_1\).

The response from those treatment combinations are represented by \(a_0b_0 \equiv (1), (a_0b_1) \equiv (b), (a_1b_0) \equiv (a)\) and \((a_1b_1) \equiv (ab)\), respectively.
Now consider the following:

I. \( \frac{(C_0V_0)+(C_0V_1)}{2} \): Average effect of voltage for current level \( C_0 \)
   \[ \frac{(a_0b_0)+(a_0b_1)}{2} \equiv \frac{1}{2} + \frac{b}{2}. \]

II. \( \frac{(C_iV_0)+(C_iV_1)}{2} \): Average effect of voltage for current level \( C_i \)
   \[ \frac{(a_i b_0)+(a_i b_1)}{2} \equiv \frac{a}{2} + \frac{(a) + (ab)}{2}. \]

Compare these two group means (or totals) as follows:

Average effect of \( V_1 \) level – Average effect at \( V_0 \) level
\[ = \frac{(a) + (ab)}{2} - \frac{1}{2} + \frac{a}{2} \]
\[ = \text{Main effect of voltage} \]
\[ = \text{Main effect of } B. \]

Comparison like

\( (C_0V_1) - (C_0V_0) \equiv (a) - (1) \) indicate the effect of voltage at current level \( C_0 \)
and

\( (C_iV_1) - (C_iV_0) \equiv (ab) - (b) \) indicate the effect of voltage at current level \( C_i. \)
The average interaction effect of voltage and current can be obtained as
\[
\begin{align*}
&\left(\text{Average effect of voltage at current level } I_0\right) - \left(\text{Average effect of voltage at current level } I_1\right) \\
= &\text{ Average effect of voltage at different levels of current.} \\
= &\frac{(C_v I_1) - (C_v I_0) - (C_v I_1) - (C_v I_0)}{2} \\
= &\frac{(ab) - (b) - (a) - (1)}{2} \\
= &\text{ Average interaction effect.}
\end{align*}
\]

Similarly,
\[
\frac{(C_v I_0) + (C_v I_0)}{2} = \frac{(1) + (b)}{2} : \text{ Average effect of current at voltage level } V_0.
\]
\[
\frac{(C_v I_1) + (C_v I_1)}{2} = \frac{(a) + (ab)}{2} : \text{ Average effect of current at voltage level } V_1.
\]

Comparison of these two as
\[
\begin{align*}
&\left(\text{Average effect of current at voltage level } V_0\right) - \left(\text{Average effect of current at voltage level } V_1\right) \\
= &\frac{(C_v I_1) + (C_v I_1) - (C_v I_0) - (C_v I_0)}{2} \\
= &\frac{(a) + (ab) - (1) + (b)}{2} \\
= &\text{ Main effect of current} \\
= &\text{ Main effect of } A.
\end{align*}
\]
Comparison like

$$(C_1V_0) - (C_0V_0) = (b) - (1): \text{Effect of current at voltage level } V_0$$

$$(C_1V_1) - (C_0V_1) = (ab) - (a): \text{Effect of current at voltage level } V_1.$$ 

The average interact effect of current and voltage can be obtained as

$$\left(\frac{\text{Average effect of current at voltage level } V_0}{\text{Average effect of current at voltage level } V_i}\right) - \left(\frac{\text{Average effect of current at different levels of voltage}}{2}\right)$$

$$= \text{Average effect of current at different levels of voltage}$$

$$= \frac{(C_1V_1) - (C_0V_1)}{2} - \frac{(C_1V_0) - (C_0V_0)}{2}$$

$$= \frac{(ab) - (a)}{2} - \frac{(b) - (1)}{2}$$

$$= \text{Average interaction effect}$$

$$= \text{Same as average effects of voltage at different levels of current.}$$

(It is expected that the interaction effect of current and voltage is same as the interaction effect of voltage and current).

The quantity

$$\frac{(C_0V_0) + (C_1V_0) + (C_0V_1) + (C_1V_1)}{4} = \frac{(1) + (a) + (b) + (ab)}{4}$$

gives the general mean effect of all the treatment combination.