2³ Factorial experiment

Suppose that in a complete factorial experiment, there are three factors - A, B and C, each at two levels, viz., \(a_0, a_1; b_0, b_1\) and \(c_0, c_1\) respectively. There are total eight number of combinations:

\[a_0b_0c_0, a_0b_0c_1, a_0b_1c_0, a_0b_1c_1, a_1b_0c_0, a_1b_0c_1, a_1b_1c_0, a_1b_1c_1.\]

Each treatment combination has \(r\) replicates, so the total number of observations are \(N = 2^3r = 8r\) that are to be analyzed for their influence on the response.

Assume the total response values are

\[Y_* = [(1), \ a, \ b, \ ab, \ c, \ ac, \ bc, \ abc]'.\]

The response values can be arranged in a three-dimensional contingency table. The effects are determined by the linear contrasts

\[l'_{\text{effect}}Y_* = l'_{\text{effect}}((1), \ a, \ b, \ ab, \ c, \ ac, \ bc, \ abc)\]

using the following table:
<table>
<thead>
<tr>
<th>Factorial effect</th>
<th>Treatment combinations</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(1)</td>
</tr>
<tr>
<td>I</td>
<td>+</td>
</tr>
<tr>
<td>A</td>
<td>-</td>
</tr>
<tr>
<td>B</td>
<td>-</td>
</tr>
<tr>
<td>AB</td>
<td>+</td>
</tr>
<tr>
<td>C</td>
<td>-</td>
</tr>
<tr>
<td>AC</td>
<td>+</td>
</tr>
<tr>
<td>BC</td>
<td>+</td>
</tr>
<tr>
<td>ABC</td>
<td>-</td>
</tr>
</tbody>
</table>
Note that once few rows have been determined in this table, rest can be obtained by simple multiplication of the symbols. For example, consider the column corresponding to \( a \), we note that \( A \) has + sign, \( B \) has – sign, so \( AB \) has – sign (= sign of \( A \times \) sign of \( B \)).

Once \( AB \) has - sign, \( C \) has – sign then \( ABC \) has (sign of \( AB \) x sign of \( C \)) which is + sign and so on.

The first row is a basic element. With this \( a = 1^tY \). can be computed where 1 is a column vector of all elements unity. If other rows are multiplied with the first row, they stay unchanged (therefore we call it as identity and denoted as \( I \)). Every other row has the same number of + and – signs. If + is replaced by 1 and – is replaced by -1, we obtain the vectors of orthogonal contrasts with the norm \( 8(=2^3) \).

If each row is multiplied by itself, we obtain \( I \) (first row). The product of any two rows leads to a different row in the table. For example

\[
A.B = AB
\]
\[
AB.B = AB^2 = A
\]
\[
AC.BC = A.C^2BB = AB.
\]

The structure in the table helps in estimating the average effect.
For example, the average effect of $A$ is

$$A = \frac{1}{4r} \left[ (a) - (1) + (ab) - (b) + (ac) - (c) + (abc) - (bc) \right]$$

which has following explanation.

(i) Average effect of $A$ at low level of $B$ and $C \equiv (a_i b_o c_o) - (a_o b_o c_o) \equiv \frac{(a) - (1)}{r}$

(ii) Average effect of $A$ at low level of $B$ and low level of $C \equiv (a_i b_i c_o) - (a_o b_o c_o) \equiv \frac{(ab) - (b)}{r}$

(iii) Average effect of $A$ at low level of $B$ and high level of $C \equiv (a_i b_i c_1) - (a_o b_o c_1) \equiv \frac{(ac) - (c)}{r}$

(iv) Average effect of $A$ at low level of $B$ and $C \equiv (a_i b_i c_1) - (a_o b_o c_1) \equiv \frac{(abc) - (bc)}{r}$.

Hence for all combinations of $B$ and $C$, the average effect of $A$ is the average of all the average effects in (i)-(iv).
Similarly, other main and interaction effects are as follows:

\[
B = \frac{1}{4r} \left[ (b) + (ab) + (bc) + (abc) - (1) - (a) - (c) - (ac) \right] = \frac{(a+1)(b-1)(c+1)}{4r}
\]

\[
C = \frac{1}{4r} \left[ c + (ac) + (bc) + (abc) - (1) - (a) - (b) - (ab) \right] = \frac{(a+1)(b+1)(c-1)}{4r}
\]

\[
AB = \frac{1}{4r} \left[ (1) + (ab) + (c) + (abc) - (a) - (b) - (ac) - (bc) \right] = \frac{(a-1)(b-1)(c+1)}{4r}
\]

\[
AC = \frac{1}{4r} \left[ (1) + (b) + (ac) + (abc) - (a) - (ab) - (c) - (bc) \right] = \frac{(a-1)(b+1)(c-1)}{4r}
\]

\[
BC = \frac{1}{4r} \left[ (1) + (a) + (bc) + (abc) - (b) - (ab) - (c) - (ac) \right] = \frac{(a+1)(b-1)(c-1)}{4r}
\]

\[
ABC = \frac{1}{4r} \left[ (abc) + (ab) + (b) + (c) - (ab) - (ac) - (bc) - (1) \right] = \frac{(a-1)(b-1)(c-1)}{4r}
\]

Various sum of squares in the $2^3$ factorial experiment are obtained as

\[
SS(Effect) = \frac{(linear\ contrast)^2}{8r} = \frac{\left( \ell^2 \right)^2}{r \ell^2 Y^2}
\]

which follow a Chi-square distribution with one degree of freedom under normality of $Y$. The corresponding mean squares are obtained as

\[
MS(Effect) = \frac{SS(Effect)}{Degrees\ of\ freedom}.
\]
The corresponding $F$-statistics are obtained by

$$F_{\text{effect}} = \frac{MS(\text{Effect})}{MS(\text{Error})}$$

which follows an $F$-distribution with $\text{df}_{\text{effect}}$ degrees of freedoms 1 and error degrees of freedom under respective null hypothesis. The decision rule is to reject the corresponding null hypothesis at $\alpha$ level of significance whenever

$$F_{\text{effect}} > F_{1-\alpha} (1, df_{\text{error}}).$$

These outcomes are presented in the following ANOVA table.

<table>
<thead>
<tr>
<th>Sources</th>
<th>Sum of squares</th>
<th>Degrees of freedom</th>
<th>Mean squares</th>
<th>$F$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>SSA</td>
<td>1</td>
<td>$MSA = SSA / 1$</td>
<td>$F_A$</td>
</tr>
<tr>
<td>$B$</td>
<td>SSB</td>
<td>1</td>
<td>$MSB = SSB / 1$</td>
<td>$F_B$</td>
</tr>
<tr>
<td>$AB$</td>
<td>SSAB</td>
<td>1</td>
<td>$MSAB = SSAB / 1$</td>
<td>$F_{AB}$</td>
</tr>
<tr>
<td>$C$</td>
<td>SSC</td>
<td>1</td>
<td>$MSC = SSC / 1$</td>
<td>$F_C$</td>
</tr>
<tr>
<td>$AC$</td>
<td>SSAC</td>
<td>1</td>
<td>$MSAC = SSAC / 1$</td>
<td>$F_{AC}$</td>
</tr>
<tr>
<td>$BC$</td>
<td>SSBC</td>
<td>1</td>
<td>$MSBC = SSBC / 1$</td>
<td>$F_{BC}$</td>
</tr>
<tr>
<td>$ABC$</td>
<td>SSABC</td>
<td>1</td>
<td>$MSABC = SSABC / 1$</td>
<td>$F_{ABC}$</td>
</tr>
<tr>
<td>Error</td>
<td>SS(Error)</td>
<td>$8(r-1)$</td>
<td>$MS(\text{Error}) = SS(\text{Error}) / {8(r-1)}$</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>TSS</td>
<td>$8r - 1$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>